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CONSTRUCTING OF MULTIPLE REFLECTIONS FROM THE INNER SUFACES OF AN SPHERE AND A RIGHT CIRCULAR CYLINDER

The goal of this paper is performing a geometrical analysis of the phenomenon of the mirror reflection from the inner sufaces of an sphere and a right circular cylinder. Such construction is sought, which would allow to realize a projection of reflection of any given point **P** situated on the projective plane $\mathbf{S^3}$ in the mirror formed on the surface of the inner sphere $\boldsymbol{\lambda_2}$ or the cylinder $\boldsymbol{\gamma_2}$**.** Such point **R** belonging to the given surface should be found, that the radius going from the point **P** having reflected in the mirror in point **R**, would go through the centre of projection **E** (eye). This problem has already been a subject of research in the field of descriptive geometry as far as direct reflections are concerned, i.e. the cases when the radius going from point **P** to **E** reflects only in point **R**. At the same time, the author is not familiar with such works which in their interest embrace also the phenomenon of multiple reflections. The radius going from point **P,** in its way to point **E**, may additionally (also many times) reflect in the concave mirror, before it reflects in **R**.

Let us consider such situation, in which points **P**, **E** and **R** are not identical, and are not collinear. In addition, we will assume, as for the sphere, that the points **P** and **E** are not identical with point **O** (the centre of the sphere), and as for the cylinder, that the points **P** and **E** do not belong to the axis of the cylinder.

Geometrical properties of the phenomenon of multiple reflections in a sphere are as follows:

1. Points **E**, **P**, **R**, **O** (fig.1) as well as all the indirect points of reflection $\mathbf{R_1}$, $\mathbf{R_2}$, ..., $\mathbf{R_n}$ belong to the same plane $\boldsymbol{\pi}$. Therefore, in the case of a sphere, the problem considered is equivalent to the problem of seeking points of reflection on the circle $\mathbf{m_2}$, which is an intersection of the sphere $\boldsymbol{\lambda_2}$ and the plane $\boldsymbol{\pi}$.
2. All segments of the broken line which describes the way of the radius from the point **P** to the eye **E** are tangent to the specific for this broken line circle $\mathbf{s_2}$ with the centre in point **O**.
3. The angles $\boldsymbol{\alpha}$ between the following segments of the broken line are congruent.
4. Adopting from the mechanics the idea of torque, we could observe that if the segments of the broken line are transformed into vectors (we will assume, that the sense of the vector will be directed duly from the point **P** in the direction of **E**), then all these vectors (including the first and the

last ones) will form the moments of force towards point **O** with the clockwise sense, or they will all have the anti-clockwise sense.

5. The angles formed with the following points of osculation between the segments of the broken line and the centre of the circle $\angle \mathbf{S_1OS_2}$, $\angle \mathbf{S_2OS_3}$, ..., $\angle \mathbf{S_{n-1}OS_n}$ are congruent. If the sum of the angles from $\angle \mathbf{S_1OS_2}$ to $\angle \mathbf{S_{n-1}OS_n}$ equals n-1 times $\angle \mathbf{S_1OS_2}$ has the magnitude δ, then the angle formed by the first point of osculation $\mathbf{S_1}$, the centre of the circle **O** and the last point of osculation $\mathbf{S_n}$ (in our case it is the angle $\boldsymbol{\beta}$) equals the magnitude δ or δ reduced by the multiple of the turn, so that the radian measure $\boldsymbol{\beta}$ is contained in the interval $(0, 2\pi)$. The angle $\boldsymbol{\beta}$ is measured from the ray $\mathbf{OS_1}$ to the ray $\mathbf{OS_n}$ in the clockwise direction or anti-clockwise, but in agreement with applying in the given case property 4.

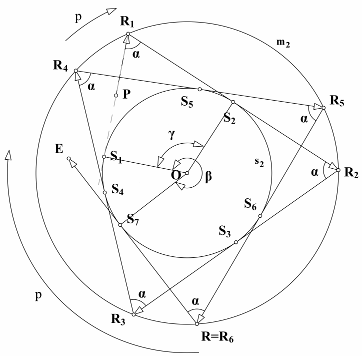

Fig. 1

On the basis of the acquainted properties of the phenomenon of multiple reflections from the circle, we can formulate a series of possible solutions to the problem (fig. 2). The point **P** can belong to two lines tangent to the circle $\mathbf{s_2}$ (2 cases), point **E** can also belong to two lines tangent to the circle $\mathbf{s_2}$ (2×2 cases), the vectors coming out of point **P** can form moments of force of "left-" or "right-handed" rotation ($2 \times 2 \times 2$ cases), the senses of the vectors ending in **E** must comply with property 4, therefore they do not alter the number of cases.

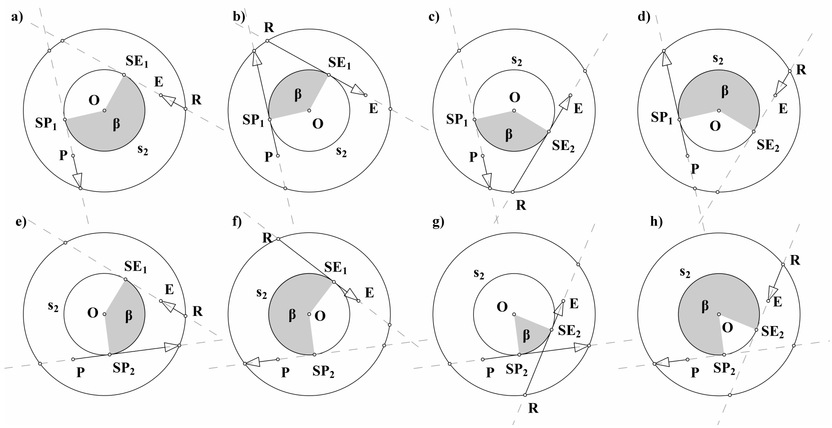

Fig.2

The last parameter which determins the potential number of possible solutions is the multiple of reflexion. If we decide that the reflexion will be multiplied by m, then for each couple of cases based on the same points of osculation, e.g. for cases a) and b) there exists m-number of potential cases. Summing up, we have $2 \times 2 \times 2/2 \times \mathrm{m}$ possible cases to solve for the fixed multiple of reflexion. For instance, there would be 16 cases of fourfold reflexion.

Using the described properties of multiple mirror reflexions from the surface of the inner sphere, we can complete an approximate construction of seeking the point of reflexion (fig. 3). The construction will be completed as an instance for one of the cases of fourfold reflexion (m=4). We have the given points **E** and **P** as well as the circle $\mathbf{m_2}$ with the centre in point **O** to which the sought point **R** belongs. We draw any given circle $\mathbf{s_2}$ with the centre in **O** and the radius equal to or less than the shorter of the segments **OE** and **OP**. We find tangents to this circle, going through **E** and **P**. We will mark the points of osculation accordingly as $\mathbf{SE_1}$, $\mathbf{SE_2}$ and $\mathbf{SP_1}$ i $\mathbf{SP_2}$. We will consider the case based on the points of osculation $\mathbf{SE_1}$ and $\mathbf{SP_1}$ and the clockwise sense of rotation. Points **P** and $\mathbf{SP_1}$ will form a straight line to which the first segment of the broken line describing the run of the radius belongs, while the points **E** and $\mathbf{SE_1}$ will form a straight line to which the last segment of the broken line belongs. To find the other straight lines which will contain the missing segments of the broken line, we will need the tangents to the circle in the points $\mathbf{S_2}$, $\mathbf{S_3}$ and $\mathbf{S_4}$. On the grounds of property 5 we know that, to find them, we must divide the angle $\boldsymbol{\beta}$**,** or the angle $\boldsymbol{\beta}$ increased by the multiply of the turn (2π), into four equal parts (in general, into m equal parts). In accordance with the property 4, the angle produced in this way must be less than the straight angle. In our example, these conditions are met both by the angle $\boldsymbol{\varphi}$ and the angle $\boldsymbol{\gamma}$, which is equal to $\boldsymbol{\varphi}$ multiplied by $\pi/2$. Generally speaking, the angle $\boldsymbol{\varphi}$ is increased by the multiply of $2\pi/m$ in the following possible cases. For our further considerations, we will choose the case in which the angles produced by the following points of osculation and the centre of circles **O** equal **γ**. In this way we obtain points $\mathbf{S_2}$, $\mathbf{S_3}$ and $\mathbf{S_4}$ on which we can already mark tangents to $\mathbf{s_2}$, and in the points of intersection of the following one another tangents, points $\mathbf{R_1}$, $\mathbf{R_2}$, $\mathbf{R_3}$, $\mathbf{R_4}$. All these points belong to the circle with the centre in **O**, we shall call it $\mathbf{k_2}$. If the sought point was the one of the fourfold reflection from the circle $\mathbf{k_2}$, then one of the solutions would be point **R***. In order to find the sought point on the circle $\mathbf{m_2}$ we should repeat the above construction for a number of circles $\mathbf{s_2}$ of different radii. The following positions of the point **R*** will form a curve $\mathbf{f_2}$, whose common point with the circle $\mathbf{m_2}$ is the solution sought in the considered example.

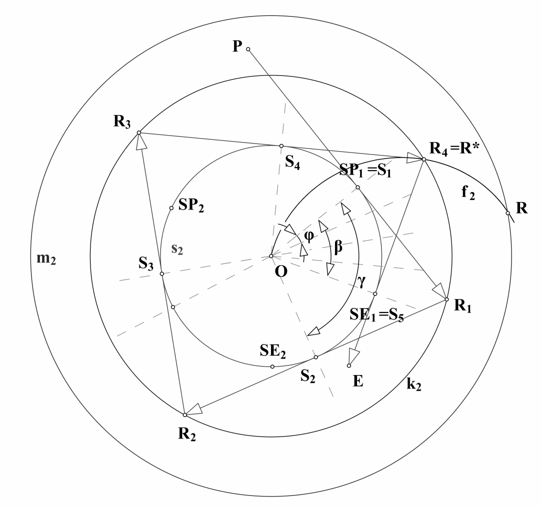

Fig.3

In the case of reflexion from the inner surface of a cylinder the following indirect points of mirror reflection would belong to a helix, but on a projective plane square to the axis of the cylinder we acquire the possibility of using the construction described for a circle (fig. 3), yet we should bear in mind, that in such a case, in points $\mathbf{R_1}$, $\mathbf{R_2}$, ..., $\mathbf{R_n}$, we obtain the images of the generating lines of the cylinder, on which a mirror reflexion is performed. In order to find the final point of reflexion, we should expand the fragments of the planes parallel to the axis of the cylinder, described on this projection by the means of segments $\mathbf{R_1R_2}$, $\mathbf{R_2R_3}$, ..., $\mathbf{R_{n-1}R_n}$. On this expansion, we obtain the points of the following reflexions on the intersections of the straight line connecting **P** with **E** and the generating lines of the cylinder.