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ABOUT A SPECIFIC CONSTRUCTION OF A CONICAL SURFACE

Let us assume a straight line l and three incollinear points A, B, C, none of which lies on line l. The task is to determine such a circular conical surface, that comprises both the line l and the points A, B, C.

It means that such a point W_x (located on the line l) must be found that straight lines produced from it through the 3 given points will form elements of this circular conical surface.

Let us assume for this purpose any point W_1 on the line l and draw straight lines from it passing through the 3 given points A, B, C. Then let us assume, at the same distance from W_1, points A_1, B_1, C_1 on these lines and a point D_1 on the line l, as well as points $A^0{}_1$, $B^0{}_1$, $C^0{}_1$, $D^0{}_1$ in such a way that point W_1 is the centre of the segments $A_1A^0{}_1$, $B_1B^0{}_1$, $C_1C^0{}_1$, $D_1D^0{}_1$. Let us further assume pairs of planes α_1 and $\alpha^0{}_1$, β_1 and $\beta^0{}_1$, γ_1 and $\gamma^0{}_1$, δ_1 and $\delta^0{}_1$, perpendicular to the straight lines $A_1A^0{}_1$, $B_1B^0{}_1$, $C_1C^0{}_1$, $D_1D^0{}_1$ respectively, where $A_1 \in \alpha_1$ and $A^0{}_1 \in \alpha^0{}_1$, $B_1 \in \beta_1$ and $B^0{}_1 \in \beta^0{}_1$, $C_1 \in \gamma_1$ and $C^0{}_1 \in \gamma^0{}_1$, $D_1 \in \delta_1$ and $D^0{}_1 \in \delta^0{}_1$, and $\alpha_1 \parallel \alpha^0{}_1$, $\beta_1 \parallel \beta^0{}_1$, $\gamma_1 \parallel \gamma^0{}_1$, $\delta_1 \parallel \delta^0{}_1$ as well as $\alpha_1 \perp A_1A^0{}_1$, $\beta_1 \perp B_1B^0{}_1$, $\gamma_1 \perp C_1C^0{}_1$, $\delta_1 \perp D_1D^0{}_1$.

In such a case planes α_1, β_1, γ_1, δ_1 and $\alpha^0{}_1$, $\beta^0{}_1$, $\gamma^0{}_1$, $\delta^0{}_1$ will form an octahedron of parallelepiped type with the centre in point W_1 of 12 vertices in which triplets of planes meet:

$1(\alpha_1, \beta_1, \gamma_1)$	$1^0(\alpha^0{}_1, \beta^0{}_1, \gamma^0{}_1)$
$2(\alpha_1, \gamma_1, \delta_1)$	$2^0(\alpha^0{}_1, \gamma^0{}_1, \delta^0{}_1)$
$3(\alpha_1, \beta_1, \gamma^0{}_1)$	$3^0(\alpha^0{}_1, \beta^0{}_1, \gamma_1)$
$4(\beta_1, \gamma^0{}_1, \delta^0{}_1)$	$4^0(\beta^0{}_1, \gamma_1, \delta_1)$
$5(\beta_1, \gamma_1, \delta^0{}_1)$	$5^0(\beta^0{}_1, \gamma^0{}_1, \delta_1)$

$6\,(\alpha^0{}_1, \gamma_1, \delta^0{}_1)$ $\qquad$ $6^0\,(\alpha_1, \gamma^0{}_1, \delta_1)$

Therefore, the position of planes $\alpha_1, \beta_1, \gamma_1, \delta_1$ and $\alpha^0{}_1, \beta^0{}_1, \gamma^0{}_1, \delta^0{}_1$ should be converted in such a way, that corresponding vertices 1 and 2, as well as 1^0 and 2^0, join each other forming a pair of opposite vertices 1_x and $1^0{}_x$ determined by quadruples of planes $1_x(\alpha_x, \beta_x, \gamma_x, \delta_x)$ and $1^0{}_x(\alpha^0{}_x, \beta^0{}_x, \gamma^0{}_x, \delta^0{}_x)$. The vertices 1_x and $1^0{}_x$ will then mark out the axis of circular surface Φ^2 with vertex W_x on line *l*, being at the same time the centre of the octahedron thus obtained.

In order to obtain the axis and vertex of the circular conical surface, let us define the following construction. Let us assume the straight line *l* and 3 arbitrary incollinear points *A*, *B*, *C*. Let us then draw, at any point *D* on the line *l*, a plane δ perpendicular to *l*. Then after assuming an arbitrary point W_1 on line l, 3 straight lines should be produced from W_1 and through points *A*, *B*, *C*. Starting from point W_1, let us measure off on these lines the W_1D segment length to obtain such 3 points A_1, B_1, C_1, that $W_1D \equiv W_1A_1 \equiv W_1B_1 \equiv W_1C_1$. In these points perpendicular planes $\alpha_1, \beta_1, \gamma_1$ should be assumed, that $\alpha \perp W_1A_1$, $\beta \perp W_1B_1$, $\gamma \perp W_1C_1$. The planes $\alpha_1, \beta_1, \gamma_1$ will form with the plane δ 3 points Z_1, Z_2, Z_3, where:

$Z_1(\alpha_1\ \beta_1\ \delta)$, $Z_2(\beta_1\ \gamma_1\ \delta)$, $Z_3(\alpha_1\ \gamma_1\ \delta)$. Subsequently assumed locations of point W_1 on line *l* are reflected by points $Z_1(\alpha_1\ \beta_1\ \delta)$, $Z_2(\beta_1\ \gamma_1\ \delta)$, $Z_3(\alpha_1\ \gamma_1\ \delta)$, which will describe curves $\{Z_1\}$, $\{Z_2\}$, $\{Z_3\}$. There exists a point W_x on line *l*, for which the four planes $\alpha_x, \beta_x, \gamma_x, \delta$ will intersect in a single point $Z_x \equiv \{Z_1\} \cap \{Z_2\} \cap \{Z_3\} \equiv Z_x(\alpha_x, \beta_x, \gamma_x, \delta)$.

Without further discussing the properties of the curves $\{Z_1\}$, $\{Z_2\}$, $\{Z_3\}$, let us notice, that line segments connecting pairs of the curves' points, that are located closest to the point Z_x, form an approximation of the lines' arcs. The intersection point thus obtained is therefore an approximation of the point Z_x being searched for.

Because in the parallelepiped-type octahedron the point Z_x can be represented by any of the two vertices $1(\alpha, \beta, \gamma, \delta)$ or $1^0(\alpha^0, \beta^0, \gamma^0, \delta^0)$ that determine the axis of a circular conical surface, then point Z_x is also the searched-for point of the circular conical surface's axis. The vertex of this surface is located at point W_x, its axis is defined by pair of points W_x, Z_x, while its element – by the given line *l*. Assuming that line and any triplet of incollinear points *A*, *B*, *C* determine circular conical surface Φ^2, the latter's elements that include the points will form, together with *l*, an arbitrary asymmetric configuration – (see Witold Szymański, *About characteristics of a specific octahedron*, case 1) – and, as the solution, only one cone Φ^2 having the vertex W_x can be found. For symmetric configuration of the elements – (see idem, case 2 and 3) – two $(\Phi_1^2 \Phi_2^2)$ or three $(\Phi_1^2 \Phi_2^2 \Phi_3^2)$ circular conical surfaces would exist and be of common vertex W_x as well as common cone's elements.

The construction discussed above, being approximate, is nevertheless sufficient for practical determination of a circular conical surface, when its element and three incollinar points are given.