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ABOUT CHARACTERISTICS OF A SPECIFIC OCTAHEDRON

Let us assume a circle k^2 on a sphere K^2. For an arbitrary point of the circle, let us assume a normal to the sphere and a plane tangent to the sphere at this point. Such a pair is mutually perpendicular. All the pairs in question assumed for points of the circle k^2 form two circular conic surfaces with a common circle k^2 and vertices lying on a common axis. One of the surfaces is tangent to the sphere K^2 at the circle k^2, its vertex being an external point with respect to the sphere, while the other surface intersects the sphere at the circle k^2 and has its vertex and at the center of the sphere.

Let us intersect four arbitrary elements *a*, *b*, *c*, *d* of a circular conical surface S^2 with a sphere K^2, the sphere's vertex being also the vertex of S^2. Next, let us assume, in points thus obtained A and A^0, B and B^0 and, C and C^0, D and D^0, of the elements *a*, *b*, *c* and *d*, pairs of plaines α and α^0, β and β^0, γ and γ^0, δ and δ^0, tangent to the sphere, where $A \in \alpha$, $A^0 \in \alpha^0$, $B \in \beta$, $B^0 \in \beta^0$, $C \in \gamma$, $C^0 \in \gamma^0$, $D \in \delta$, $D^0 \in \delta^0$. As a result, pairs of plaines $\alpha \parallel \alpha^0$, $\beta \parallel \beta^0$, $\gamma \parallel \gamma^0$, $\delta \parallel \delta^0$ arise, where $\alpha \perp AA^0$, $\beta \perp BB^0$, $\gamma \perp CC^0$, $\delta \perp CC^0$. Therefore, the planes $\alpha, \beta, \gamma, \delta$ intersect at the point *1* lying on the axis of the conical circular surface S^2, while the planes α^0, β^0, γ^0, δ^0 intersect at the point 1^0 of this axis, respectively.

Let us introduce the fallowing definition: the set of eight planes α, α^0, β, β^0, γ, γ^0, δ, δ^0, parallel in pairs $\alpha \parallel \alpha^0$, $\beta \parallel \beta^0$, $\gamma \parallel \gamma^0$, $\delta \parallel \delta^0$ and equally distant from the point *W*, forms an parallelepiped-type octahedron. Point *W* forms the center of the octahedron. Pairs of parallel planes forms 4 pairs of opposite faces.

To determine the of concept parallelepiped-type octahedron more clearly, let us notice, that its faces planes can be treated as tangents to the sphere K^2.

Let us now consider possible configurations of points on the sphere, for which the sphere is tangent to the planes $\alpha, \beta, \gamma, \delta, \alpha^0, \beta^0, \gamma^0, \delta^0$ of the octahedron's faces. Points of tangency form 4 pairs situated symmetrically with respect to the sphere's center, thus being the opposite points of the sphere. Let us designate pairs of opposite points as $A, A^0, B, B^0, C, C^0, D, D^0$. Points A, B, C, D, as well their corresponding opposite points A^0, B^0, C^0, D^0, form one of 4 layouts of points on the sphere.

Layouts 1, 2, 3 are planar layouts out of all that can be formed by these points. Because points A, B, C, D and A^0, B^0, C^0, D^0 are equally distant form the center of the sphere K^2, the planes determined by them are equidistant from the sphere's center and mutually parallel. The planes intersect the sphere in two congruent and circle k^2 and k^{2o}, comprising points A, B, C, D and A^0, B^0, C^0, D^0, respectively.

The configuration of points A, B, C, D and A^0, B^0, C^0, D^0 on the circles k^2 and k^{2o}, can be:

Case 1: arbitrary, i.e. nonsymmetric

Case 2: symmetric with respect to a single symmetry axis

Case 3: symmetric with respect to two symmetry axes

The fourth possible configuration of points exists for:

Case 4: such an arbitrary position of points A, B, C, D and A^0, B^0, C^0, D^0 on the sphere, that they form no coplanar quadruples.

For the first three cases, if quadruples of points A, B, C, D and A^0, B^0, C^0, D^0 form planar layouts, then the direct lines determined by pairs of points $A A^0, B B^0, C C^0, D D^0$ are elements of a circular conical surface S^2. For the case 2 and 3, they are also elements of one or two more circular conical surfaces P^2 and Q^2, different from the assumed surface S^2. Because quadruples of planes $\alpha, \beta, \gamma, \delta$ and $\alpha^0, \beta^0, \gamma^0, \delta^0$ intersect on the axis of the surface S^2 in points *1* and 1^0, then quadruples of planes from among all the eight ones $\alpha, \beta, \gamma, \delta, \alpha^0, \beta^0, \gamma^0, \delta^0$ will intersect on the axes of the surfaces P^2 and Q^2, as they have common elements with the surface S^2.

Let us now make up a specification that characterizes parallelepiped-type octahedrons formed by planes $\alpha, \beta, \gamma, \delta, \alpha^0, \beta^0, \gamma^0, \delta^0$ with respect to the cases discussed above.

For the case 1, the octahedron has two opposite vertices *1* and 1^0, that are formed by quadruples of planes, 4 pairs of opposite vertices, that are formed by triples of planes, and 16 edges. Vertices *1* and 1^0 determine an axis *s* of the circular surface S^2. Vertices *1* and 1^0 form the planes:

$1\,(\alpha, \beta, \gamma, \delta)$ and $1^0(\alpha^0, \beta^0, \gamma^0, \delta^0)$.

For the case 2, the octahedron has two pairs of opposite vertices *1* and 1^0, *2* and 2^0, in which quadruples of planes intersect, 2 pairs of opposite vertices, that are formed by triplets of planes, and 14 edges. Vertices *1* and 1^0, *2* and 2^0 determine axes *s* and *p* of two circular surfaces S^2 and P^2. Vertices *1* and 1^0, *2* and 2^0 form the following planes:

$1\,(\alpha,\ \beta,\ \gamma,\ \delta)$ and $1^0\,(\alpha^0,\ \beta^0, \gamma^0, \delta^0)$,

$2\,(\alpha,\ \beta,\ \gamma,\ \delta)$ and $2^0\,(\alpha^0,\ \beta^0, \gamma^0, \delta^0)$.

For the case 3, the octahedron has three pairs of opposite vertices *1* and 1^0, *2* and 2^0, *3* and 3^0, determine mutually perpendicular axes *s*, *p* and *q* of three circular conical surfaces S^2, P^2 and Q^2. Vertices *1* and 1^0, *2* and 2^0, *3* and 3^0 form the following planes:

$1\,(\alpha,\ \beta,\ \gamma,\ \delta)$ and $1^0\,(\alpha^0,\ \beta^0, \gamma^0, \delta^0)$,

$2\,(\alpha,\ \beta,\ \gamma,\ \delta)$ and $2^0\,(\alpha^0,\ \beta^0, \gamma^0, \delta^0)$,

$3\,(\alpha,\ \beta,\ \gamma,\ \delta)$ and $3^0\,(\alpha^0,\ \beta^0, \gamma^0, \delta^0)$.

For the case 4, i.e. when points *A*, *B*, *C*, *D* and A^0, B^0, C^0, D^0 are arbitrary points of the sphere K^2, the straight lines $A\,A^0$, $B\,B^0$, $C\,C^0$, $D\,D^0$ are not elements of a circular conical surface. The octahedron has then 6 pairs of vertices formed by triplets of plaines, and 18 edges.

As a result of the discussion above, the concept of a parallelepiped-type octahedron has been determined. Four possible configurations have been discussed of points of tangency of the sphere with respect to the octahedron's faces. In relation to these configurations of points, four sample drawings of the octahedrons have been constructed, having 0, 1, 2 or 3 pairs opposite vertices, in which quadruples of opposite faces intersect. It has been shown, that pairs of the vertices determine axes of 1, 2 or 3 circular conical surfaces with their vertex in the center of the octahedron. If 2 or 3 conical surfaces exist, they have 4 common elements. The elements run through the points of tangency of the octahedron with the sphere, the octahedron is circumscribed on. The paper *About a specific construction of a conical surface* will present an example of employing relationships found here between a parallelepiped-type octahedron and a circular conical surface.