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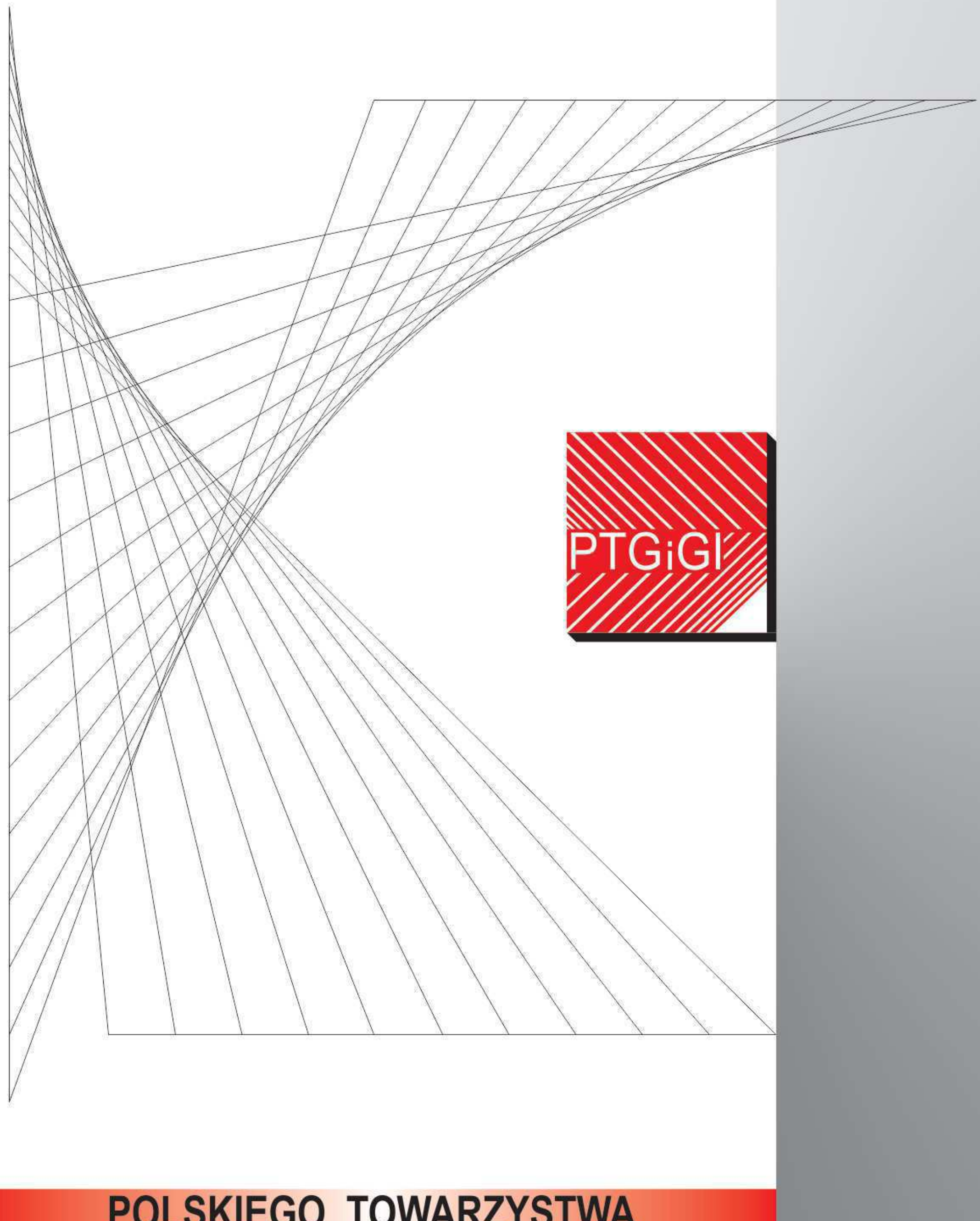

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ISOMETRIES IN TEACHING DESCRIPTIVE GEOMETRY AND ENGINEERING GRAPHICS

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Abstract. Important basic transformations, implemented in CAD systems, are congruence transformations, so-called isometries, which preserve the distance of points. Logic of CAD software bases on the reflection, translation, rotation, and similarity. This fact is the important desideratum in the teaching of Descriptive Geometry. The paper includes a proposal for a teaching from the scope of isometries on the plane and in three-dimensional space.

Keywords: reflection, symmetry about a plane, central symmetry, translation, rotation, isometry, descriptive geometry, tessellation, Platonic solid

1 Introduction

Important basic transformations, implemented in CAD systems, are congruence transformations, so-called *isometries*, which preserve the distance of points. Logic of CAD software bases on the *reflection*, *translation*, and *rotation*. Slightly more general are *similarity* transformations (also implemented in CAD), which change the distance of points in the same ratio. These transformations allow to construct more complex objects on the base of constructed previous (earlier) simple *elementary objects* (so-called *primitives*).

2 Isometries

As we said before, isometries are the transformations of plane (E^2) or generally *n*–dimensional spaces (E^n) which preserve the distance of points. The basic (primitive) transformation is the symmetry S_H across the (n–1) – dimensional hyperplane H in E^n.

2.1 Isometries in E^2

Isometries on the plane E^2 have a set of generators the collections of reflections (axial symmetries). Every isometry is
1: reflection (*axial symmetry*) S_x
or
2: the composition of two reflections S_a, S_b,
which is
A: the translation $\mathsf{T}_{2AB} = \mathsf{S}_b\mathsf{S}_a$, if the axes a, b are parallel a||b (Fig. 1)
or
B: the rotation $\mathsf{R}_{O,2\varphi} = \mathsf{S}_b\mathsf{S}_a$, if the axes a, b intersect $a \cap b = \{O\}$ (Fig. 2).
or
3: the composition of three reflections S_a, S_b, S_c (Fig. 4).
If $a \perp b$, then $\mathsf{R}_{O,2\cdot 90} = \mathsf{S}_b\mathsf{S}_a$, and we have the central symmetry $\mathsf{S}_O = \mathsf{S}_b\mathsf{S}_a$ [1].
The composition of three reflections is symmetry or glide reflection (glide symmetry). A composition of two reflections is direct transformation (not change the orientation of a space), a reflection or a composition of three reflections is opposite transformation (change

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the orientation of a space). Therefore translations and rotations are direct transformations. So, for example: triangles PQR and S_x(PQR) are oppositely congruent, and triangles PQR and S_yS_x(PQR) are directly congruent for every no colinear points P, Q, R, and lines x, y.

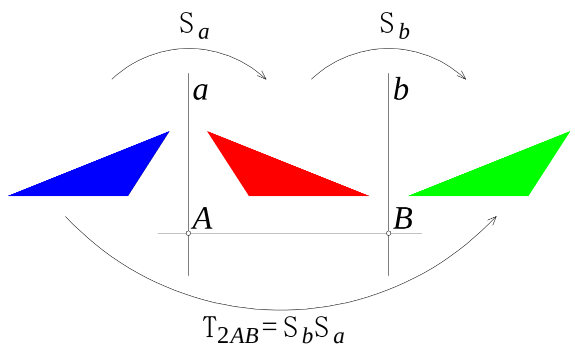

Figure 1: The composition S_bS_a of the reflections (axial symmetries) S_a, S_b across equal a=b or parallel lines $a \| b$ is a translation T_{2AB} (AB – the vector determined by the points A, B)

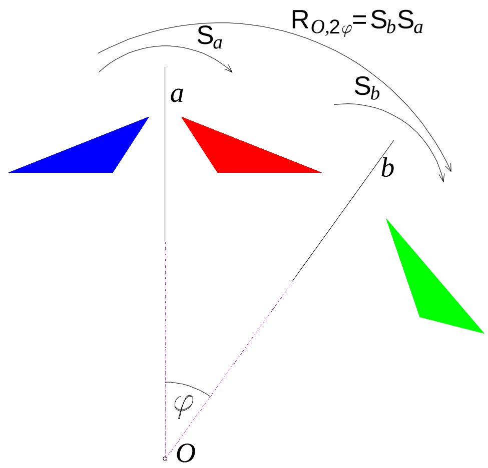

Figure 2: The composition S_bS_a of the reflections (axial symmetries) S_a, S_b across intersecting $a \cap b$ = {O} lines (axes) a, b is a rotation $R_{O,2\varphi}$

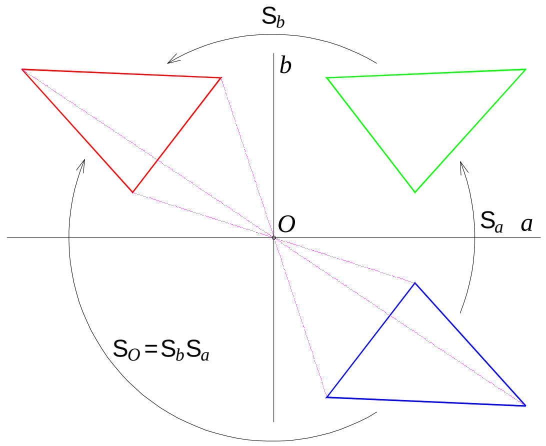

Figure 3: The composition S_bS_a of two reflections S_a, S_b across the perpendicular lines a, b ($a \perp b$) is the central symmetry S_O

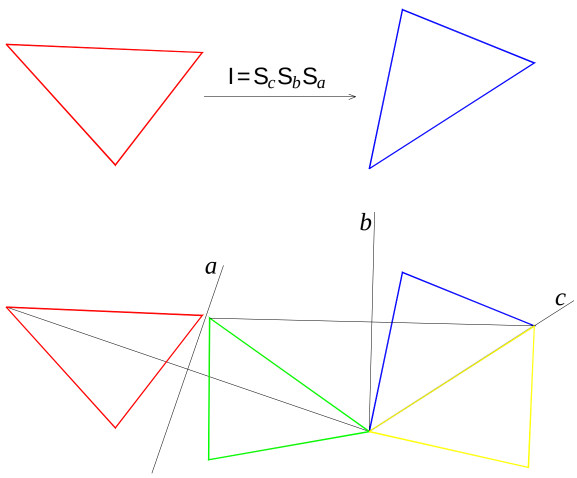

Figure 4: The composition $S_cS_bS_a$ of three reflections S_a, S_b , S_c across the lines a, b, c mapping the left side triangle onto the right side triangle. At the bottom: given left side triangle and right side triangle

Two triangles with equal side lengths we call *congruent* triangles. For two congruent triangles it exist an isometry such that the first of them can be mapped into second one (Fig. 1÷4). To realize these transformation we need at most three reflections (Fig. 4). Two congruent triangles can be mapped into each other either by a direct or an opposite isometry (Fig. 1÷4).

2.1.1 2.1.1 Tessellations in E^2 – some examples

Example 2.1.1: Transformations on a parallelogram. We take two arbitrary curves and translate them in two directions (Fig. 5). We obtain the elementary object ("pigeon") (Fig. 5).

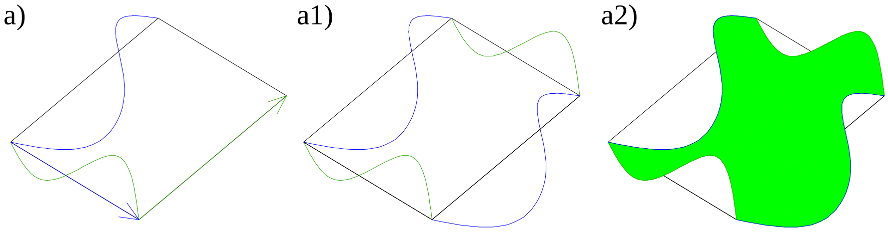

Figure 5: Transformations on a parallelogram: a) two arbitrary curves; a1) translation images of them; a2) obtained shape ("pigeon") [3]

Further translations in the same directions conduit to a certain tessellation (Fig. 6).

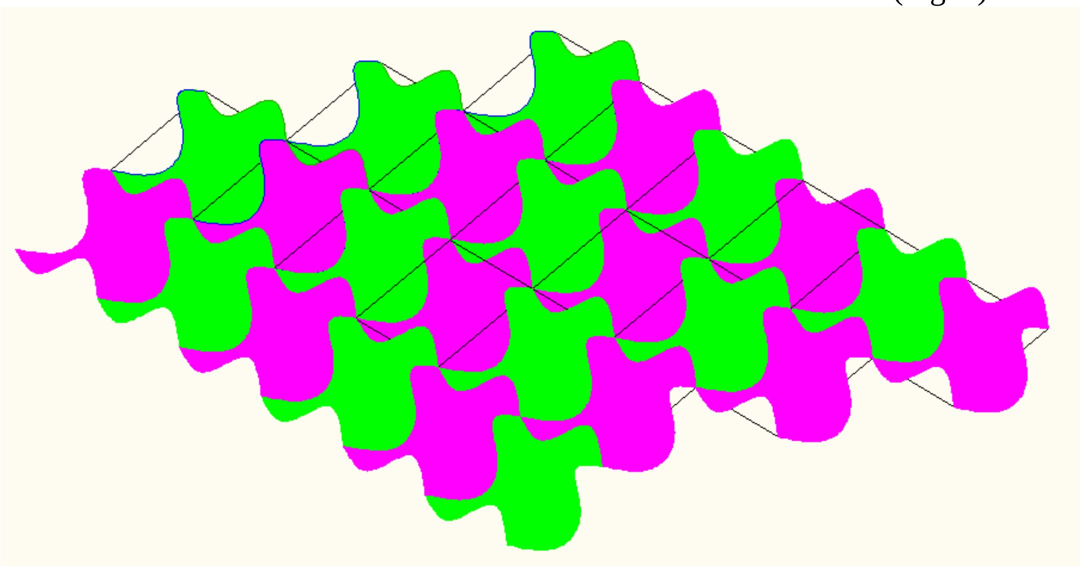

Figure 6: Parallelogram two-curved tessellation ("pigeons") [3]

Example 2.1.2: Transformations on a equilateral triangle**.** We take an arbitrary curve with the origin in a vertex and endpoint in the middle of a side of the given equilateral triangle (Fig. 7a). Then we transform it by the central symmetry (Fig. 7a1), next by two rotations (Fig. 7a2), and by five rotations (Fig. 7a3).

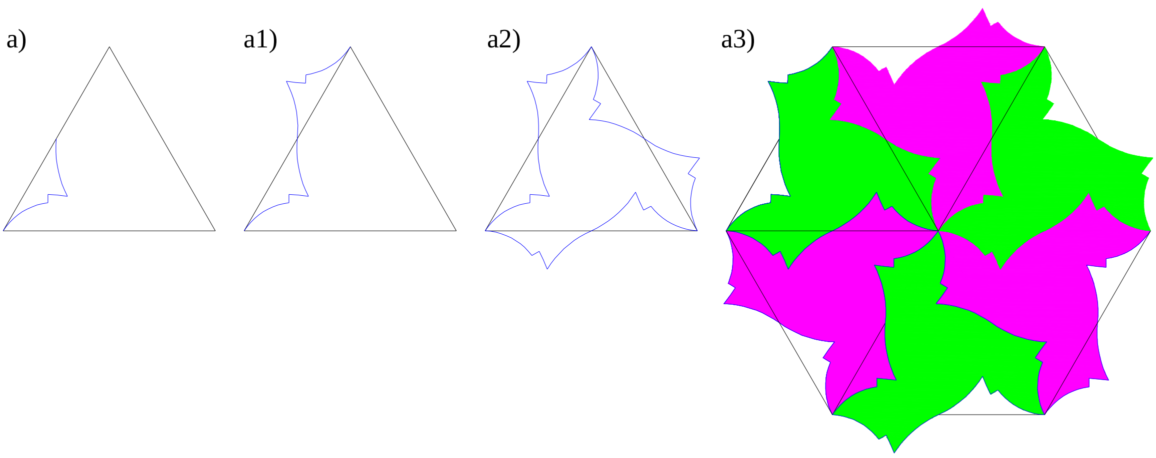

Figure 7: Transformations on a equilateral triangle – variations on Escher's three butterflies: a) arbitrary curve; a1) central symmetry image of it; a2) two rotations; a3) six (five) rotations [3]

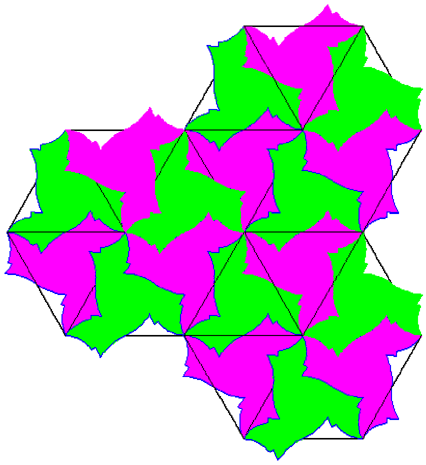

Figure 8: Two rotations of the shape from Figure 7 [3]

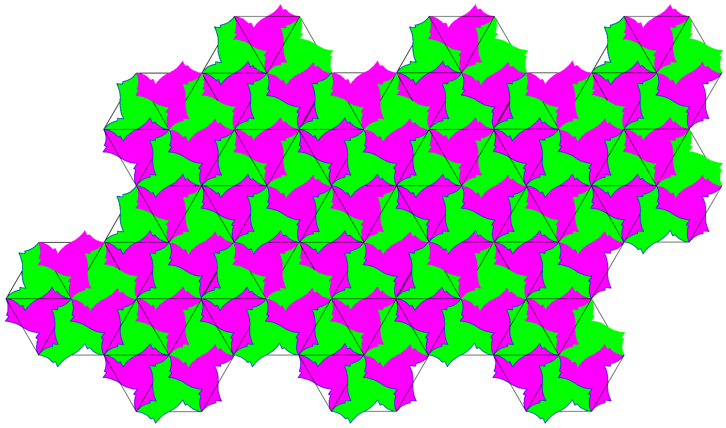

Figure 9: Some appropriate translations of the shape from Figure 8 in directions parallel to the sides of triangles (hexagons) [3]

Example 2.1.3: Two arbitrary curves with the origin in a vertex and endpoint in the second vertex of a side of on given square (Fig. 10a). Then we turn such curves around the appropriate vertices with the rotation angle 90^{o} and we obtain the shape of a lizard (Fig. 10a1).

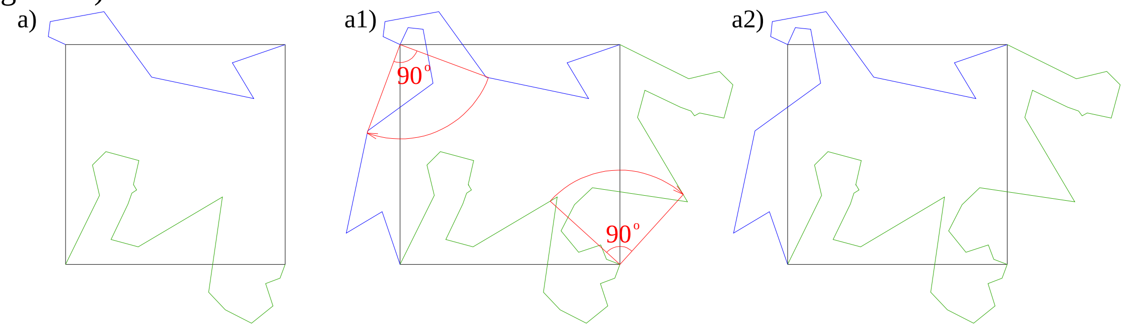

Figure 10: Transformations on a square: a) two arbitrary curves; a1) two rotations around the vertices with a rotation angle 90^{o}; a2) the obtained elementary shape [3]

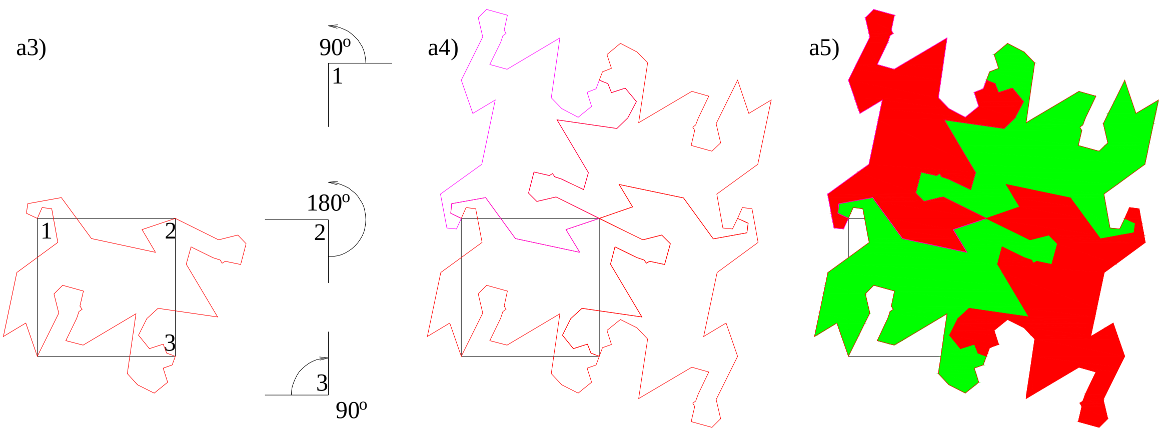

Figure 11: Transformations on a square: a3) three distinguished vertices; a4) the configuration obtained by three rotations around the vertices with appropriate rotation angles 90^{o}, 180^{o}, 90^{o}; a5) the obtained complex shape [3]

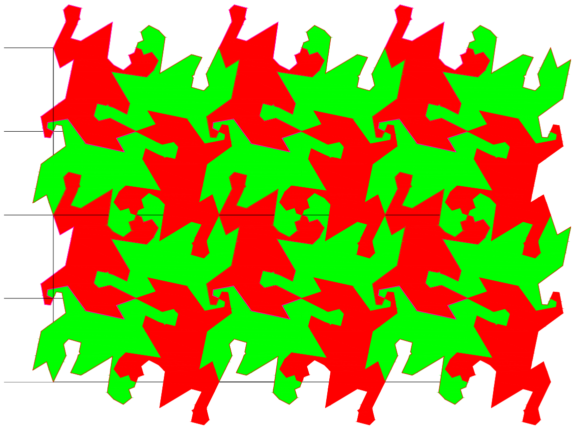

Figure 12: Transformations on a square – the obtained tessellation [3]

Figure 13: Concrete paving designed on the equilateral triangle basis (photo: E. Koźniewski)

Interesting information about the tessellations can be found in many publications, including [1, 4]. All tessellations are described by seventeen discrete groups of isometries involving two independent translations [1, 4]. It is noteworthy that six of the seventeen two-dimensional space groups arise as the symmetry groups of familiar patterns of rectangles, which we may think of as bricks or tiles [1].

2.2 Isometries in E^3

As we said before, reflection, translation, and rotation in (two-dimensional) plane can be generalized to three dimensions. One generalization of a two-dimensional reflection is three-dimensional *reflection* S_x *about a straight line x*, called the *reflection line*. We can consider also an another *reflection* S_ω *about a plane* ω, called the *reflection plane*. Isometries on the plane E^3 have a set of generators the collections of reflections about a plane. Every isometry is

1: *reflection* about a plane S_α

or

2: the composition of two reflections S_α, S_β,

which is

A: the *translation* $\mathsf{T}_{2AB} = \mathsf{S}_\beta\mathsf{S}_\alpha$, if the reflection planes α, β are parallel $\alpha \| \beta$

or

B: the *rotation* $\mathsf{R}_{c,2\varphi} = \mathsf{S}_\beta\mathsf{S}_\alpha$, if the reflection planes α, β intersect $\alpha \cap \beta = c$; for $\varphi = 90^{\circ}$ we have reflection S_c about the straight line c

or

3: the composition of three reflections S_α, S_β, S_γ,

which is

A: the *glide reflection* (a glide reflection is a composition a reflection about a plane and a translation parallel to that plane)

or

B: the *rotation* with the orthogonal reflection about a plane,

4: the composition of four reflections S_α, S_β, S_γ, S_δ which is the *helical motion* [1].

The isometries 2 and 4 are direct isometries, the isometries 1 and 3 are opposite. Spatial congruence transformations (isometries) are an essential tool when modeling objects.

Example 2.2.1: The modeling of two platonic polyhedrons: dodecahedron and icosahedron. First we construct the (horizontal) wall of a dodecahedron, next using Monge method we find the slope angle of five walls [2] and rotate the horizontal wall to obtain the first inclined wall (Fig. 14a), then we rotate four times the inclined wall (Fig. 14a1), transform by a symmetry about the plane (Fig. 14a2), rotate and move to the position in Figure 14a3.

a) a1) a2) a3)

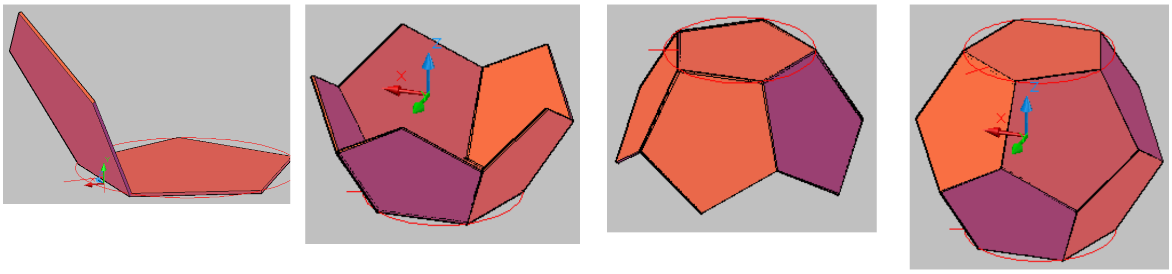

Figure 14: Creating the model of regular dodecahedron: a) finding (by means the Monge method) the appropriate angle; a1) five (four) rotations; a2) reflection about a plane; a3) one rotation and one translation [2]

A similar design, but solid, for the icosahedron we can see in Figure 15 [2].

a) a1) a2) a3) a4) a5)

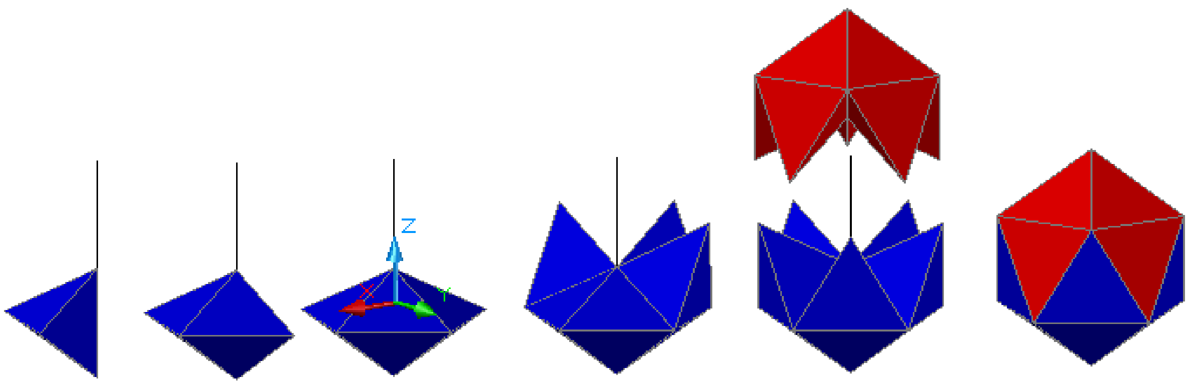

Figure 15: Creating the model of regular icosahedron: a) finding (by means the Monge method) the appropriate triangular pyramid with a ridge in horizontal position; a1) one reflection about the wall of pyramid; a2) three reflections about appropriate planes or three rotations ; a3) three reflections about appropriate walls; a4) two consecutive reflections and one reflection about the plane determined by five top vertices of the bottom solid, and one appropriate rotation about the vertical line; a5) and one translation [2]

Example 2.2.2: The modeling of a *rhombic dodecahedron*. A rhombic dodecahedron can be generated by adding congruent pyramid to the six walls of a cube (Fig. 16). First we construct the quadrangular pyramid with the height equals a half the cube edge. The further construction shows Figure 16. A rhombic dodecahedron has interesting properties: it fills the space without gapes (Fig. 17). This spatial filling operation resembles a tessellation on a plane. Using translation in two directions, we can fill the entire three-dimensional Euclidean space. Rhombic dodecahedron can be treated as a sophisticated brick-shaped.

a) a1) a2) a3) a4)

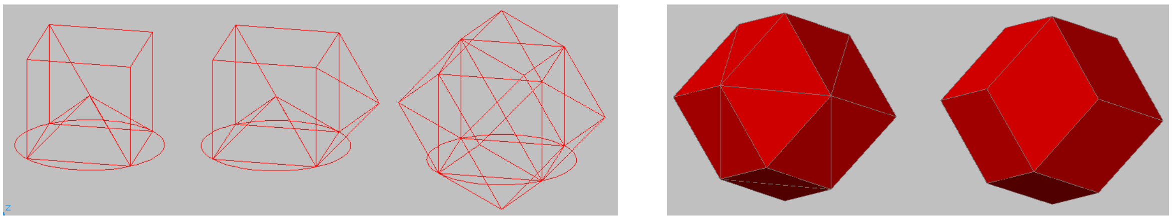

Figure 16: Creating the model of a rhombic dodecahedron: a) construction the quadrangular pyramid with the height equals a half the cube edge; a1) one rotation; a2) one reflection, three rotation and one translation; a3) the complete solid with visual decomposition; a4) the obtained solid after the Boolean operation – sum [3]

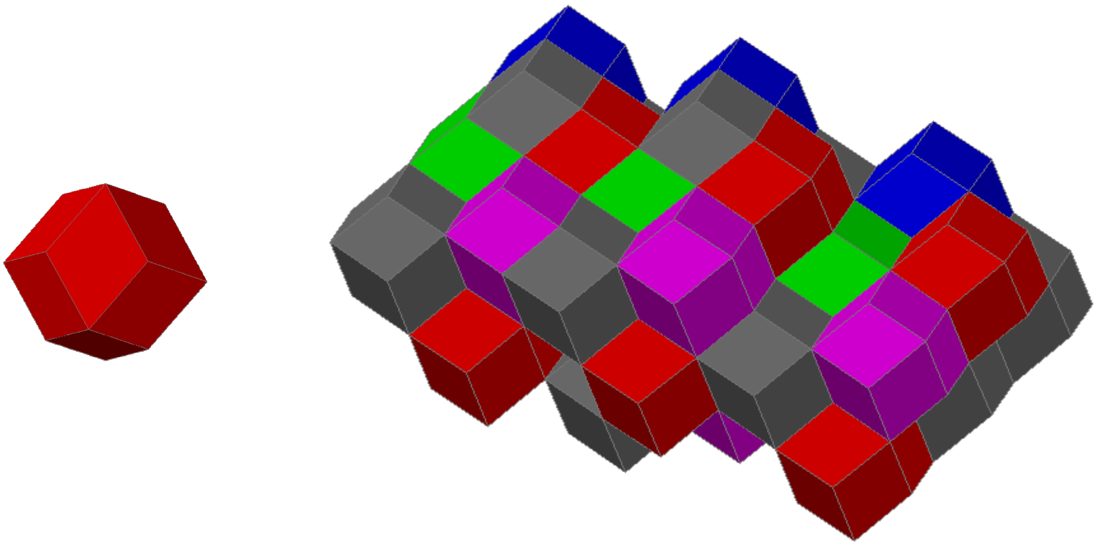

Figure 17: Rhombic dodecahedron can be treated as a sophisticated brick-shaped [3]

Example 2.2.3: The virtual construction of a model of Art Tower by Arata Isozaki [4]. The model is obtained by the skillful use of reflection about a plane to the previously constructed tetrahedron. There are the reflections about selected faces of given tetrahedron.

a) a1) a2) a3)

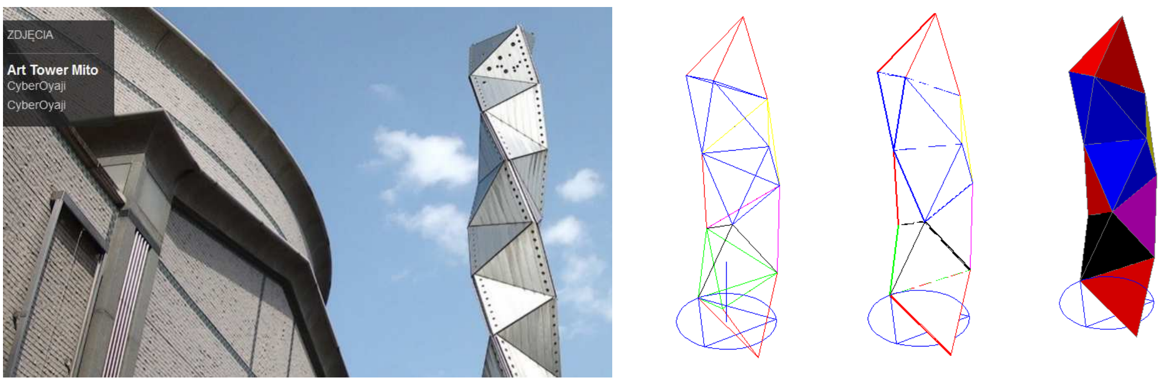

Figure 18: Creating the model of *Art Tower*: a) the *Art Tower* in Mito by Arata Isozaki [4]; a1÷a3) the sequence of reflection tetrahedrons in various visualizations [3]

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3 Conclusions

Construction of regular solids in three-dimensional space and design tessellation on a plane are excellent situations application of concepts of symmetry. This is a good field of a creative exploration for the student.

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IZOMETRIE W NAUCZANIU GEOMETRII WYKREŚLNEJ I GRAFIKI INŻYNIERSKIEJ

Ważnymi przekształceniami zaimplementowanymi w oprogramowaniu CAD, są izometrie, czyli przekształcenia zachowujące odległość punktów. Logika tych systemów opiera się gównie na pojęciu symetrii, translacji i obrotu. Ważną jeszcze rolę odgrywa podobieństwo. Uwzględnienie tego faktu w nauczaniu geometrii wykreślnej jest ważnym dezyderatem dydaktycznym. Praca zawiera propozycję dydaktyczną z zakresu zastosowania izometrii na płaszczyźnie i w przestrzeni.