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ISOMETRIES IN TEACHING DESCRIPTIVE GEOMETRY AND ENGINEERING GRAPHICS

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Abstract. Important basic transformations, implemented in CAD systems, are congruence transformations, so-called isometries, which preserve the distance of points. Logic of CAD software bases on the reflection, translation, rotation, and similarity. This fact is the important desideratum in the teaching of Descriptive Geometry. The paper includes a proposal for a teaching from the scope of isometries on the plane and in three-dimensional space.

Keywords: reflection, symmetry about a plane, central symmetry, translation, rotation, isometry, descriptive geometry, tessellation, Platonic solid

1 Introduction

Important basic transformations, implemented in CAD systems, are congruence transformations, so-called isometries, which preserve the distance of points. Logic of CAD software bases on the reflection, translation, and rotation. Slightly more general are similarity transformations (also implemented in CAD), which change the distance of points in the same ratio. These transformations allow to construct more complex objects on the base of constructed previous (earlier) simple elementary objects (so-called primitives).

2 Isometries

As we said before, isometries are the transformations of plane ($E^2$) or generally $n$–dimensional spaces ($E^n$) which preserve the distance of points. The basic (primitive) transformation is the symmetry $S_H$ across the $(n–1)$ – dimensional hyperplane $H$ in $E^n$.

2.1 Isometries in $E^2$

Isometries on the plane $E^2$ have a set of generators the collections of reflections (axial symmetries). Every isometry is

1: reflection (axial symmetry) $S_x$
or
2: the composition of two reflections $S_a, S_b$, which is
   A: the translation $T_{2AB}= S_b S_a$, if the axes a, b are parallel $a\|b$ (Fig. 1)or
   B: the rotation $R_{O,2\phi} = S_b S_a$, if the axes a, b intersect $a\cap b = \{O\}$ (Fig. 2).
or
3: the composition of three reflections $S_a, S_b, S_c$ (Fig. 4).

If $a\perp b$, then $R_{O,2\cdot90} = S_b S_a$, and we have the central symmetry $S_O = S_b S_a$ [1].

The composition of three reflections is symmetry or glide reflection (glide symmetry). A composition of two reflections is direct transformation (not change the orientation of a space), a reflection or a composition of three reflections is opposite transformation (change
the orientation of a space). Therefore translations and rotations are direct transformations. So, for example: triangles PQR and \( S_x(PQR) \) are oppositely congruent, and triangles PQR and \( S_xS_y(PQR) \) are directly congruent for every no colinear points P, Q, R, and lines x, y.

Two triangles with equal side lengths we call **congruent** triangles. For two congruent triangles it exist an isometry such that the first of them can be mapped into second one (Fig. 1–4). To realize these transformation we need at most three reflections (Fig. 4). Two congruent triangles can be mapped into each other either by a direct or an opposite isometry (Fig. 1–4).

2.1.1 Tessellations in \( \mathbb{E}^2 \) – some examples

Example 2.1.1: Transformations on a parallelogram. We take two arbitrary curves and translate them in two directions (Fig. 5). We obtain the elementary object ("pigeon") (Fig. 5).
Figure 5: Transformations on a parallelogram: a) two arbitrary curves; a1) translation images of them; a2) obtained shape (“pigeon”) [3]

Further translations in the same directions conduit to a certain tessellation (Fig. 6).

Figure 6: Parallelogram two-curved tessellation (“pigeons”) [3]

Example 2.1.2: Transformations on an equilateral triangle. We take an arbitrary curve with the origin in a vertex and endpoint in the middle of a side of the given equilateral triangle (Fig. 7a). Then we transform it by the central symmetry (Fig. 7a1), next by two rotations (Fig. 7a2), and by five rotations (Fig. 7a3).

Figure 7: Transformations on an equilateral triangle – variations on Escher’s three butterflies: a) arbitrary curve; a1) central symmetry image of it; a2) two rotations; a3) six (five) rotations [3]
Example 2.1.3: Two arbitrary curves with the origin in a vertex and endpoint in the second vertex of a side of an given square (Fig. 10a). Then we turn such curves around the appropriate vertices with the rotation angle $90^\circ$ and we obtain the shape of a lizard (Fig. 10a1).

Figure 10: Transformations on a square: a) two arbitrary curves; a1) two rotations around the vertices with a rotation angle $90^\circ$; a2) the obtained elementary shape [3]

Figure 11: Transformations on a square: a3) three distinguished vertices; a4) the configuration obtained by three rotations around the vertices with appropriate rotation angles $90^\circ$, $180^\circ$, $90^\circ$; a5) the obtained complex shape [3]
Interesting information about the tessellations can be found in many publications, including [1, 4]. All tessellations are described by seventeen discrete groups of isometries involving two independent translations [1, 4]. It is noteworthy that six of the seventeen two-dimensional space groups arise as the symmetry groups of familiar patterns of rectangles, which we may think of as bricks or tiles [1].

2.2 Isometries in $\mathbb{E}^3$

As we said before, reflection, translation, and rotation in (two-dimensional) plane can be generalized to three dimensions. One generalization of a two-dimensional reflection is three-dimensional reflection $S_x$ about a straight line $x$, called the reflection line. We can consider also another reflection $S_\omega$ about a plane $\omega$, called the reflection plane. Isometries on the plane $\mathbb{E}^3$ have a set of generators the collections of reflections about a plane. Every isometry is

1: reflection about a plane $S_\alpha$
or
2: the composition of two reflections $S_\alpha S_\beta$, which is
   A: the translation $T_{2\beta_\alpha} = S_\beta S_\alpha$, if the reflection planes $\alpha, \beta$ are parallel $\alpha \parallel \beta$
or   B: the rotation $R_{\omega, 2\phi} = S_\beta S_\alpha$ if the reflection planes $\alpha, \beta$ intersect $\alpha \cap \beta = c$; for $\phi = 90^\circ$ we have reflection $S_c$ about the straight line $c$
or
3: the composition of three reflections $S_\alpha S_\beta S_\gamma$, which is
   A: the glide reflection (a glide reflection is a composition a reflection about a plane and a translation parallel to that plane)
or   B: the rotation with the orthogonal reflection about a plane,
4: the composition of four reflections $S_\alpha S_\beta S_\gamma S_\delta$ which is the helical motion [1].

The isometries 2 and 4 are direct isometries, the isometries 1 and 3 are opposite. Spatial congruence transformations (isometries) are an essential tool when modeling objects.

Example 2.2.1: The modeling of two platonic polyhedrons: dodecahedron and icosahedron. First we construct the (horizontal) wall of a dodecahedron, next using Monge method we find...
the slope angle of five walls [2] and rotate the horizontal wall to obtain the first inclined wall (Fig. 14a), then we rotate four times the inclined wall (Fig. 14a1), transform by a symmetry about the plane (Fig. 14a2), rotate and move to the position in Figure 14a3.

Figure 14: Creating the model of regular dodecahedron: a) finding (by means the Monge method) the appropriate angle; a1) five (four) rotations; a2) reflection about a plane; a3) one rotation and one translation [2]

A similar design, but solid, for the icosahedron we can see in Figure 15 [2].

Figure 15: Creating the model of regular icosahedron: a) finding (by means the Monge method) the appropriate triangular pyramid with a ridge in horizontal position; a1) one reflection about the wall of pyramid; a2) three reflections about appropriate planes or three rotations; a3) three reflections about appropriate walls; a4) two consecutive reflections and one reflection about the plane determined by five top vertices of the bottom solid, and one appropriate rotation about the vertical line; a5) and one translation [2]

Example 2.2.2: The modeling of a rhombic dodecahedron. A rhombic dodecahedron can be generated by adding congruent pyramid to the six walls of a cube (Fig. 16). First we construct the quadrangular pyramid with the height equals a half the cube edge. The further construction shows Figure 16. A rhombic dodecahedron has interesting properties: it fills the space without gapses (Fig. 17). This spatial filling operation resembles a tessellation on a plane. Using translation in two directions, we can fill the entire three-dimensional Euclidean space. Rhombic dodecahedron can be treated as a sophisticated brick-shaped.
Figure 16: Creating the model of a rhombic dodecahedron: a) construction the quadrangular pyramid with the height equals a half the cube edge; a1) one rotation; a2) one reflection, three rotation and one translation; a3) the complete solid with visual decomposition; a4) the obtained solid after the Boolean operation – sum [3]

Figure 17: Rhombic dodecahedron can be treated as a sophisticated brick-shaped [3]

Example 2.2.3: The virtual construction of a model of Art Tower by Arata Isozaki [4]. The model is obtained by the skillful use of reflection about a plane to the previously constructed tetrahedron. There are the reflections about selected faces of given tetrahedron.

Figure 18: Creating the model of Art Tower: a) the Art Tower in Mito by Arata Isozaki [4]; a1:a3) the sequence of reflection tetrahedrons in various visualizations [3]
3 Conclusions

Construction of regular solids in three-dimensional space and design tessellation on a plane are excellent situations application of concepts of symmetry. This is a good field of a creative exploration for the student.

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IZOMETRIE W NAUCZANIU GEOMETRII WYKREŚLNEJ I GRAFIKI INŻYNIERSKIEJ

Ważnymi przekształceniami zaimplementowanymi w oprogramowaniu CAD, są izometrie, czyli przekształcenia zachowujące odległość punktów. Logika tych systemów opiera się głównie na pojęciu symetrii, translacji i obrotu. Ważną jeszcze rolę odgrywa podobieństwo. Uwzględnienie tego faktu w nauczaniu geometrii wykreślnej jest ważnym dezyderatem dydaktycznym. Praca zawiera propozycję dydaktyczną z zakresu zastosowania izometrii na płaszczyźnie i w przestrzeni.