THE JOURNAL BIULETYN
OF POLISH SOCIETY
FOR GEOMETRY AND ENGINEERING GRAPHICS

PTGiGI

POLSKIEGO TOWARZYSTWA
GEOMETRII I GRAFIKI INŻYNIERSKIEJ

VOLUME 28 / JUNE 2016

THE JOURNAL OF POLISH SOCIETY FOR GEOMETRY AND ENGINEERING GRAPHICS

VOLUME 28

Gliwice, June 2016

All the papers in this journal have been reviewed

Editorial office address:
44-100 Gliwice, ul. Krzywoustego 7, POLAND
phone: (+48 32) 237 26 58

Bank account of PTGiGI : Lukas Bank 94 1940 1076 3058 1799 0000 0000

ISSN 1644 - 9363

Publication date: June 2016 Circulation: 100 issues.

Retail price: 15 PLN (4 EU)

CONTENTS

ISSN 1644-9363 / PLN 15.00 © 2016 PTGiGI

THICKNESS ANALYSIS OF A SADDLE

Marcin KOŹNIEWSKI

Bialystok University of Technology, Faculty of Computer Science
ul. Wiejska 45A, 15-351 Białystok, POLAND
e-mail: m.kozniewski@pb.edu.pl

Abstract. The study analyzes the thickness of the solid-surface lobe saddle formed by two geometric surface parabolic hyperboloids as a result of the translation. The obtained results demonstrate the dependence between the thickness of the saddle and the ratio of height and base edge length of the base cuboid. With some values of that ratio the thickness of the saddle fit in acceptable deviations from the initial value.

Keywords: ruled surface, saddle surface, saddle-shaped roof, thickness of thick-walled sheet surface, offset surface

1 Introduction

In Poland, there are two popular implementations of roof coverings in the shape of the surface of the saddle: the overlap train station in Warszawa Ochota (Fig.1) and the roof over the entrance to the Provincial Office in Kielce (Fig.2). From the geometric point of view, these are examples of ruled and doubly curved shells [2, 4, 5, 8]. The reinforced concrete design of such a covering requires to the arrange straight steel rebars in a form of the saddle [1,8]. The simplest construction of the saddle is based on two skew diagonal of the respective side walls of the cuboid connected with straight lines. The simplest way to create a doubly curved ceiling of a given thickness, and then designing the reinforcement structure is the parallel shift of the surface of a feature vector of length q (Fig. 5). Thus the ***question arises***: what is the thickness of the resulting roof covering? Obviously the thickness is less than or equal to q. The surface layer having a thickness equal to q would yield a result of modeling the surface of an offset with respect to the output surface of the saddle [7]. However, creating such a surface will be a problem in the process of designing the appropriate formwork. Sticking with previously proposed model (parallel shift of the surface of the vector of length q) would be the best solution, assuming that we take the appropriate proportions of the cuboid modeling lobe.

The problem of the thickness of the double curved surface was discussed in the works [1, 2, 3, 4, 6, 8]. However, does not analyzed the thickness of the surface obtained as a result of the shifting geometrical surface.

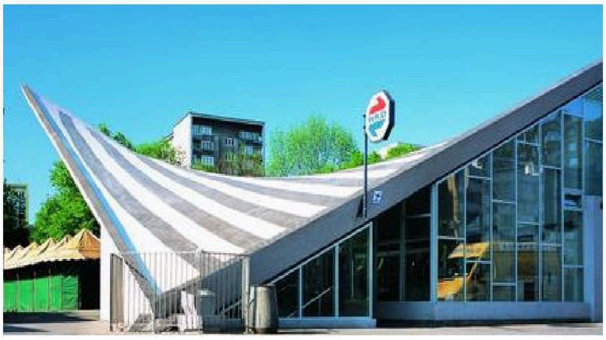

Figure 1: Warszawa Ochota train station in Warsaw – the roof over the waiting hall

Figure 2: Provincial Office in Kielce – the entry

The *hyperbolic paraboloid* (*hypar* [4, 6]) or *saddle* is one of the nine real quadric surfaces and one of the six which are ruled [4, 5]. In fact it is one of the three doubly ruled surfaces (besides the plane and the hyperboloid), having two distinct independent families of lines generating the surface.

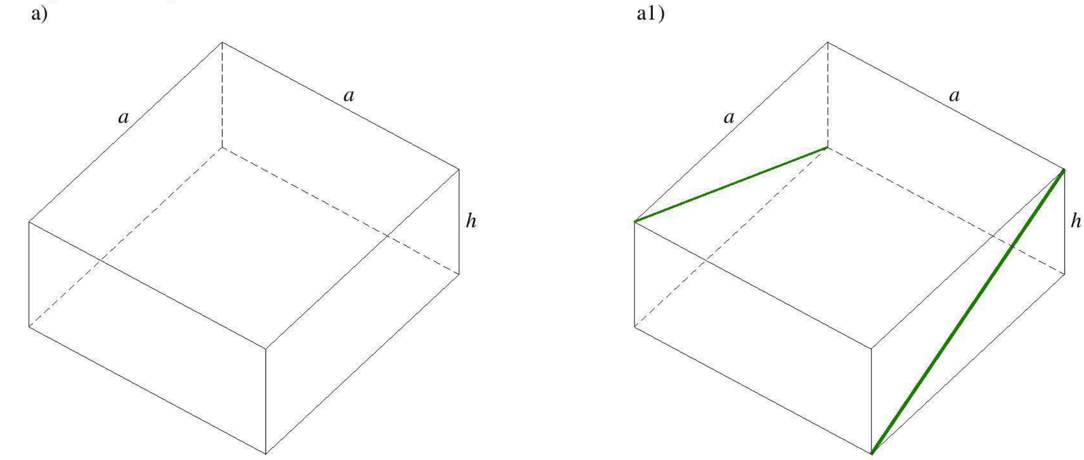

Figure 3: The construction of saddle: a) cuboid $a \times a \times h$, a1) two skew diagonals of parallel faces

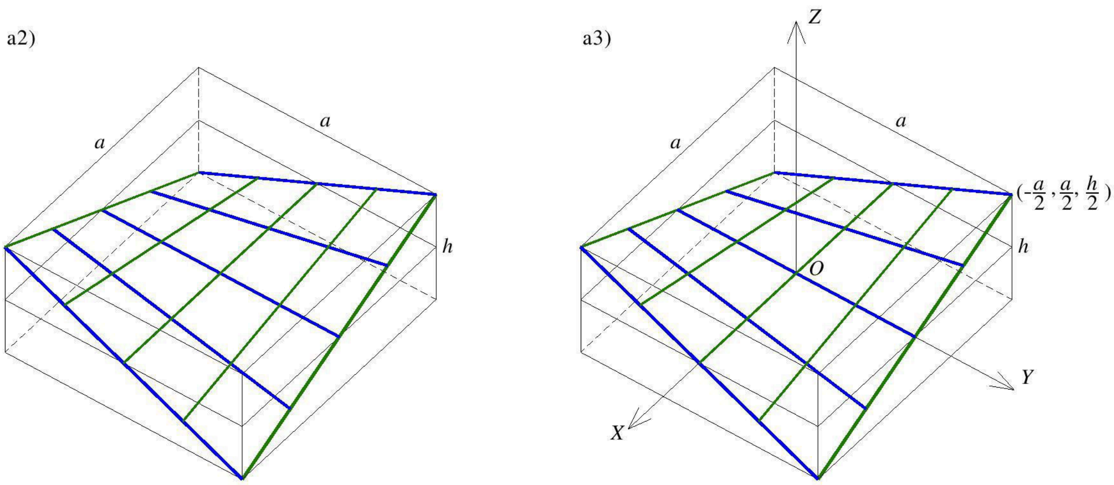

Figure 4: The saddle spanned over square $a \times a$, inscribed in cuboid $a \times a \times h$. The surface is *ruled* and has two families of linear generatrices. The equation of such a saddle is $z=kxy$

Hyperbolic paraboloid shells are structurally efficient and has many constructional and aesthetic advantages: they are used to cover large spans, vast roofed areas, and variety of other roof coverings. They are used as foundations for special structures; they can be prefabricated simply. The theoretical tools for the membrane and bending analysis of saddle shells were prepared in [3,4].

2 Geometrical characterization

Geometrically hyperbolic paraboloid we can obtain from a set of straight lines intersecting given three skew straight lines, including one at infinity. The construction of the saddle surface may be obtained considering a base cuboid of dimensions a, b, h. Without limiting the generality of considerations much you can take $a = b$. This model was adopted in these considerations. After the adoption of the relevant coordinate system (Fig. 4a3) we obtain a simple equation of the surface of the saddle:

$$z = -\frac{2h}{a^2}xy .$$

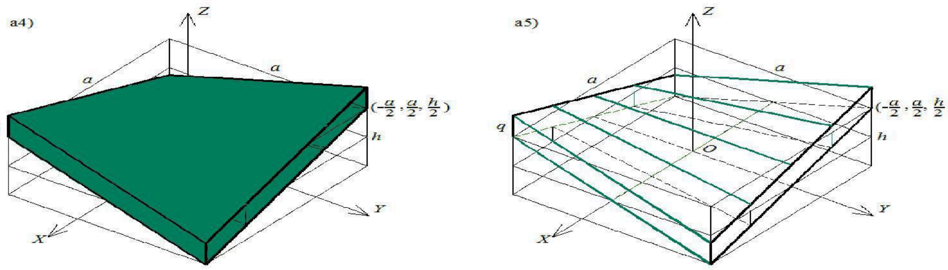

Figure 5: The saddle spanned over square $a \times a$, inscribed in prism $a \times a \times h$. The real structure, having a thickness is obtained by translation by the vector q perpendicular to the base of prism, with length q. In the coordinate system OXY there is q=[0,0,q]

3 Offset surface of the saddle

To obtain a patch of surface-trailer having a perfect thickness q should construct a surface offset. Unfortunately, the surface offset parabolic hyperboloid is no longer the surface of the saddle. Therefore, to investigate the deviation of the thickness of the panel surface of the saddle (obtained by the offset) from the surface of the offset we will conduct the following analysis.

Consider the surface

$$\mathsf{P}: \begin{cases} x_1 = x(u,v), \\ y_1 = y(u,v), \\ z_1 = z(u,v), \end{cases} \quad (u,v) \in \Delta \text{ (or where } \Delta \text{ is a rectangle: } u \in [u_1, u_2],\ v \in [v_1, v_2]),$$

which can be written as the vector function
$\mathbf{r}(u,v)=[x(u,v), y(u,v), z(u,v)]$.

3.1 Equation of Tangent Plane to a Surface at a Point

If $P_0 = (x_0, y_0, z_0)$ is a fixed point of the surface P, $P=(x, y, z)$ – any point of the tangent plane τ to the surface P in the point $P_0 = (x_0, y_0, z_0)$, P_0P is a vector $[x - x_0, y - y_0, z - z_0]$, then the equation of the tangent plane is in the form

$$\tau\colon\ \mathrm{r}'_u(u_0,v_0) \times \mathrm{r}'_v(u_0,v_0) \cdot P_0P = 0,$$

where the symbol '×' means the vector product of vectors, '·' means the scalar product of vectors. The vector $\mathbf{n} = \mathrm{r}_u^{\cdot}(u_0, v_0) \times \mathrm{r}_v^{\cdot}(u_0, v_0)$ is a normal vector to the surface P. Unit normal vector is expressed by the formula $\mathbf{n}_{\mathrm{ver}} = \dfrac{\mathrm{r}_u^{\cdot}(u_0, v_0) \times \mathrm{r}_v^{\cdot}(u_0, v_0)}{\left|\mathrm{r}_u^{\cdot}(u_0, v_0) \times \mathrm{r}_v^{\cdot}(u_0, v_0)\right|}$.

Because

$$\mathrm{r}_u^{\cdot}(u_0, v_0) = \left[\frac{\partial x}{\partial u}(u_0, v_0), \quad \frac{\partial y}{\partial u}(u_0, v_0), \quad \frac{\partial z}{\partial u}(u_0, v_0)\right]$$

and

$$\mathrm{r}_v^{\cdot}(u_0, v_0) = \left[\frac{\partial x}{\partial v}(u_0, v_0), \quad \frac{\partial y}{\partial v}(u_0, v_0), \quad \frac{\partial z}{\partial v}(u_0, v_0)\right],$$

the vector product has the form

$\mathrm{r}_u^{\cdot}(u_0, v_0) \times \mathrm{r}_v^{\cdot}(u_0, v_0) =$

$$\left[\begin{vmatrix} \frac{\partial y}{\partial u}(u_0,v_0) & \frac{\partial z}{\partial u}(u_0,v_0) \\ \frac{\partial y}{\partial v}(u_0,v_0) & \frac{\partial z}{\partial v}(u_0,v_0) \end{vmatrix} \quad \begin{vmatrix} \frac{\partial z}{\partial u}(u_0,v_0) & \frac{\partial x}{\partial u}(u_0,v_0) \\ \frac{\partial z}{\partial v}(u_0,v_0) & \frac{\partial x}{\partial v}(u_0,v_0) \end{vmatrix} \quad \begin{vmatrix} \frac{\partial x}{\partial u}(u_0,v_0) & \frac{\partial y}{\partial u}(u_0,v_0) \\ \frac{\partial x}{\partial v}(u_0,v_0) & \frac{\partial y}{\partial v}(u_0,v_0) \end{vmatrix}\right].$$

The length of the vector is expressed in the form

$\left|\mathrm{r}_u^{\cdot}(u_0, v_0) \times \mathrm{r}_v^{\cdot}(u_0, v_0)\right| =$

$$\sqrt{\begin{vmatrix} \frac{\partial y}{\partial u}(u_0,v_0) & \frac{\partial z}{\partial u}(u_0,v_0) \\ \frac{\partial y}{\partial v}(u_0,v_0) & \frac{\partial z}{\partial v}(u_0,v_0) \end{vmatrix}^2 + \begin{vmatrix} \frac{\partial z}{\partial u}(u_0,v_0) & \frac{\partial x}{\partial u}(u_0,v_0) \\ \frac{\partial z}{\partial v}(u_0,v_0) & \frac{\partial x}{\partial v}(u_0,v_0) \end{vmatrix}^2 + \begin{vmatrix} \frac{\partial x}{\partial u}(u_0,v_0) & \frac{\partial y}{\partial u}(u_0,v_0) \\ \frac{\partial x}{\partial v}(u_0,v_0) & \frac{\partial y}{\partial v}(u_0,v_0) \end{vmatrix}^2}.$$

3.2 Analysis of the thickness of the saddle

The equation of the saddle S in Figure 1a1 with the coordinate system *OXY* is

$$z = kxy. \tag{1}$$

It is easy to see that the point $\left(-\frac{a}{2}, \frac{a}{2}, \frac{h}{2}\right)$ belongs to S. So, we have the equation

$$z = -\frac{2h}{a^2}xy. \tag{2}$$

Consider the surface (2) and shifted by the vector $[0,0,q]$ the surface S_q

$$z = -\frac{2h}{a^2}xy + q. \tag{3}$$

Both these two surfaces S and S_q form a solid SS_q. More precisely, the solid SS_q can be defined as

$$\mathsf{SS}_\mathrm{q} = \left\{(x, y, z):\ (x, y) \in R \times R \text{ and } -\frac{2h}{a^2}xy \le z \le -\frac{2h}{a^2}xy + q\right\}. \tag{4}$$

There is a thick-walled sheet surface saddle SS_q. To the limited solid SS_q we will come back later. Next we will determine the equation of an offset surface for the saddle $z = -\frac{2h}{a^2}xy$. Let's write the above formulas of the surface described by a function $z=f(x,y)$, $(x,y) \in \mathrm{D}$. Parametric equations of this surface have the form $x=u$, $y=v$, $z=f(u,v)$. Further, we can write

$$\mathrm{r}'_u(u_0,v_0)=\left[\frac{\partial x}{\partial u}(u_0,v_0),\quad \frac{\partial y}{\partial u}(u_0,v_0),\quad \frac{\partial z}{\partial u}(u_0,v_0)\right]=\left[1,\quad 0,\quad \frac{\partial f}{\partial u}(u_0,v_0)\right],$$

$$\mathrm{r}'_v(u_0,v_0)=\left[\frac{\partial x}{\partial v}(u_0,v_0),\quad \frac{\partial y}{\partial v}(u_0,v_0),\quad \frac{\partial z}{\partial v}(u_0,v_0)\right]=\left[0,\quad 1,\quad \frac{\partial f}{\partial v}(u_0,v_0)\right].$$

Normal vector is expressed by the formula

$$\mathbf{n}=\left[-\frac{\partial f}{\partial u}(u_0,v_0),\quad -\frac{\partial f}{\partial v}(u_0,v_0),\quad 1\right].$$

We can go back to the notations of coordinates $u=x$, $v=y$. We obtain $\mathbf{n}=\left[-\frac{\partial f}{\partial x}(x_0,y_0),\quad -\frac{\partial f}{\partial y}(x_0,y_0),\quad 1\right]$ or $\mathbf{n}=\left[-f'_x(x_0,y_0),\quad -f'_y(x_0,y_0),\quad 1\right]$ in other form. For $f(x,y)=-\frac{2h}{a^2}xy$ we have $f'_x=-\frac{2h}{a^2}y$, $f'_y=-\frac{2h}{a^2}x$.

Then $\mathbf{n}=\left[\frac{2h}{a^2}y,\quad \frac{2h}{a^2}x,\quad 1\right]$.

The length of the vector $\mathbf{n}$ equals $|\mathbf{n}|=\sqrt{\frac{4h^2}{a^4}y^2+\frac{4h^2}{a^4}x^2+1}$.

The unit vector $\mathbf{n}_{ver}$ we can write in the form

$$\mathbf{n}_{ver}=\frac{1}{\sqrt{\left(\frac{2h}{a^2}y\right)^2+\left(\frac{2h}{a^2}y\right)^2+1}}\left[\frac{2h}{a^2}y,\quad \frac{2h}{a^2}x,\quad 1\right]$$

Parametric equations of an offset surface S_{off} with distance d for saddle $f(x,y)=-\frac{2h}{a^2}xy$ we can write in the following form:

$$f_{off}(x,y;d)\colon [X,Y,Z]=\left[x,\ y,-\frac{2h}{a^2}xy\right]+\frac{1}{\sqrt{\left(\frac{2h}{a^2}y\right)^2+\left(\frac{2h}{a^2}y\right)^2+1}}\left[d\frac{2h}{a^2}y,\quad d\frac{2h}{a^2}x,\quad d\right] \tag{5}$$

To determine the thickness of saddle SS_q limited to the square

$$\left\{(x,y):-\frac{a}{2}\le x\le\frac{a}{2},-\frac{a}{2}\le y\le\frac{a}{2}\right\} \tag{6}$$

we write the parametric equations of normal line for the point of the first surface $z=-\frac{2h}{a^2}xy$ (Fig. 4, 5). There are

$$Q\colon\begin{cases} X = x + t\cdot\left(\frac{2h}{a^2}y\right) \\ Y = y + t\cdot\left(\frac{2h}{a^2}x\right), t\in R. \\ Z = -\frac{2h}{a^2}xy + t \end{cases} \tag{7}$$

First point (belonging to surface (1)) has the coordinates $P=\left(x,y,-\frac{2h}{a^2}xy\right)$. The coordinates of point Q we find after solution of equation

$$-\frac{2h}{a^2}xy+t=-\frac{2h}{a^2}\left(x+t\frac{2h}{a^2}y\right)\left(y+t\frac{2h}{a^2}x\right)+q \tag{8}$$

Let b=$-\dfrac{2h}{a^2}$. Then equation (5) can be rewritten in the form

$bxy + t = b(x-byt)(y-bxt) +q \leftrightarrow bxy + t = b(xy - bx^2t - by^2t + b^2xyt^2) +q \leftrightarrow$
$\leftrightarrow bxy + t = bxy - b^2x^2t - b^2y^2t + b^3xyt^2 +q \leftrightarrow b^3xyt^2 + (-b^2x^2 - b^2y^2 -1)t+q=0.$

We get the equation

$$b^3xyt^2 - (b^2(x^2 + y^2) +1)t + q = 0. \tag{9}$$

We have two cases:

i: x=0 or y=0,
ii: $x\neq0$ and $y\neq0$.

Ad i: For x=0 or y=0 from (6) we get

$$t=\frac{q}{b^2(x^2+y^2)+1}. \tag{10}$$

Ad ii: Let $x\neq0$ and $y\neq0$.

We have the quadratic equation with real coefficients, that can have either one or two distinct real roots, or two distinct complex roots. In this case the discriminant determines the number and the nature of the roots. Let us compute the discriminant Δ:

$$\Delta = (-(b^2(x^2 + y^2)+1))^2 - 4b^3xyq. \tag{11}$$

Note that $b<0$. If $xy < 0$, then $\Delta > 0$.

For $xy > 0$ we run the following reasoning. We can assume that $q < 0.1a$ and $h < a$. From the structural design building viewpoint such conditions are natural. Then we can write

$$hq < \frac{1}{10}a^2 \leftrightarrow \frac{4hq}{a^2} < \frac{4}{10} < 2. \tag{12}$$

Multiplying both sides of the inequality (9) by $-xy$ we get

$$-\frac{4hq}{a^2}xy > -2xy. \tag{13}$$

Adding to both sides of the inequality (13) the expression $x^2 + y^2$ we get

$$x^2 + y^2 - \frac{4hq}{a^2}xy > x^2 + y^2 -2xy =(x- y)^2 \geq 0. \tag{14}$$

Returning to the substitution of $b = -\dfrac{2h}{a^2}$ we get

$$x^2 + y^2 -2bqxy > x^2 + y^2 -2xy =(x- y)^2 \geq 0. \tag{15}$$

Let us rewrite the expression (11) as follows

$$\Delta = b^4(x^2 + y^2)^2 + 2b^2(x^2 + y^2)+1 - 4b^3xyq = b^4(x^2 + y^2)^2 + 2b^2(x^2 + y^2 - 2bxyq) + 1. \tag{16}$$

Thus Δ>0. Let $c = \sqrt{\Delta}$. We have two solutions

$$t_1=\frac{b^2(x^2+y^2)+1-c}{2b^3xy},\ t_2=\frac{b^2(x^2+y^2)+1+c}{2b^3xy}. \tag{17}$$

The coordinates of point Q are

$$Q:\begin{cases} X = x + t_1\cdot\left(\frac{2h}{a^2}y\right) \\ Y = y + t_1\cdot\left(\frac{2h}{a^2}x\right) \\ Z = -\frac{2h}{a^2}xy + t_1 \end{cases} \tag{18.1}$$

or

$$Q:\begin{cases} X = x + t_2 \cdot \left(\dfrac{2h}{a^2} y\right) \\ Y = y + t_2 \cdot \left(\dfrac{2h}{a^2} x\right). \\ Z = -\dfrac{2h}{a^2} xy + t_2 \end{cases} \tag{18.2}$$

Making the appropriate calculations we get the difference between the panel thickness SS_q (the distance *PQ*) and panel thickness limited by surfaces S i S_{off} , which in every point equals $d(=q)$. We compiled the results in Table 1.

Table 1: The difference between the panel thickness SS_q (the distance *PQ*) and panel thickness limited by surfaces S i S_{off} , which in every point equals $d(=q)$

	a=12[m]					
No	*h* [m]	*q* [m]	*h/a*	*Thickness PQ* [m]	*PQ/q*	*PQ/q*·100%
1	2	0,4	0,166	0,389223622	0,973059056	97
2	2	0,3	0,166	0,291937919	0,973126396	97
3	2	0,2	0,166	0,194638751	0,973193754	97
4	3	0,4	0,250	0,376813737	0,942034342	94
5	3	0,3	0,250	0,282668333	0,942227778	94
6	3	0,2	0,250	0,188484275	0,942421373	94
7	4	0,4	0,333	0,361217543	0,903043858	90
8	4	0,3	0,333	0,271024645	0,903415482	90
9	4	0,2	0,333	0,180757544	0,903787718	90
10	6	0,4	0,500	0,325397886	0,813494715	81
11	6	0,3	0,500	0,244272314	0,814241048	81
12	6	0,2	0,500	0,162998025	0,814990125	81

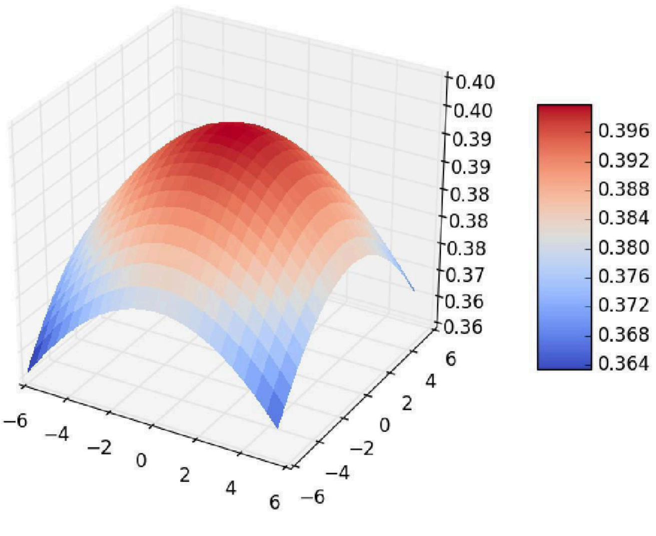

Figure 6: 3D visualization of the distribution of thickness of the saddle for *a*=12[m], *h*=4[m], *q*=0,4[m] (the plot produced with a Python script)

4 Conclusions

The analysis of the distance *PQ* allows to formulate the results. The deviation of the thickness of saddle SS_q of the thickness of the offset surface S depends on the ratio h/a (Table 1), but the deviation does not depend on the size of the translation vector (the value *q*). For h/a≤1,66 the thickness deviation of the saddle SS_q is less than 3%. For 1,66≤h/a≤0.5 the deviation (of the thickness of saddle SS_q of the thickness of the offset surface S) ranges from 3% to 19%. The exact thickness deviation (of saddle SS_q of the thickness of the offset surface S) distribution can be found in the table. These requests may be of interest to designers and contractors of complex building covers.

References

[1] Billington D. P.: *Thin Shell Concrete Structures*. McGraw-Hill Book Co.,N.Y., Revised edition, 1982.
[2] Borusiewicz W.: *Konstrukcje budowlane dla architektów*. Wydawnictwo "Arkady", Warszawa, 1978.
[3] Farshad M.: *Shell Structures*. Shiraz University Publications, Vol. I, 1986, Vol. II, 1987.
[4] Farshad M.: *Design and Analysis Shell Structures*. Kluwer, Dordrecht, The Nederlandes, 1992.
[5] Grochowski B.: *Geometria wykreślna z perspektywą stosowaną*. Wydawnictwo Naukowe PWN, Warszawa, 2016.
[6] Ortlepp R., Weiland S., Curbach M.: *Rehabilitation and Strengthening of Hypar Concrete Shell by Textile Reinforced Concrete*. Excellence in Concrete Construction through Innovation – Limbachiya & Kew (eds). 2009 Taylor & Francis Group, London.
[7] Pottmann H., Asperl A., Hofer M. and Kilian A.: *Architectural Geometry*. Bentley Institute Press, 2007.
[8] Ramaswamy G. S.: *Design and Construction of Concrete Shell Roofs*, McGraw-Hill Book Co., N.Y., 1968.
[9] Salvadori M., Heller R.: *Structure in Architecture*. Prentice-Hall, INC., N.Y., 1963.

ANALIZA GRUBOŚCI PŁATA SIODŁOWEGO

W pracy dokonano analizy grubości płata-bryły powierzchni siodłowej powstałego w wyniku przesunięcia geometrycznej powierzchni hiperboloidy parabolicznej o dany wektor. Otrzymane rezultaty wskazują na istnienie zależności grubości od proporcji wysokości i długości krawędzi podstawy prostopadłościanu bazowego. Przy pewnych proporcjach, grubości te mieszczą się w dopuszczalnych granicach odchyleń od założonej grubości płata.