INVOLUTION IN THE PENCILS OF OSCULATING CONICS $p^2_{1=2=3,4}$ AND SUPER OSCULATING CONICS $p^2_{1=2=3=4}$

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Abstract. The authors present the results of the further discussion on the properties of the pencils of the osculating and superosculating conics. Two theorems on the involutory pencils of osculating and supersculating conics and the theorem on the involutory range of points of the second order have been shown. Certain properties and construction of the basic elements of conics have been demonstrated.

Keywords: projective geometry, pencils of conics, quadratic transformation E

1 Introduction

The research work presented here is a continuation of the earlier discussion on the so–called **E** transformation which is an elation in the projective 2–D space. In the earlier publications [5], [6] and [7] the author(s) discussed the properties of the quadratic transformation **E** for which the circle $\mathbf{n}^2$ constitutes the base and the point **W** that lies on the circle $\mathbf{n}^2$ is the centre of elation. It has been shown that all lines not passing through the three points **1=2=3** coinciding with **W** point will be transformed into a set of osculating conics. The centers of these conics also create a conic which has been called the conic of centers $\mathbf{s}^2$. In the quadratic transformation **E** a parabola corresponds to the line that is tangent to the circle $\mathbf{n}^2$, a hyperbola corresponds to the line that is a secant of the circle $\mathbf{n}^2$, while an ellipse corresponds to the line that does not meet the circle $\mathbf{n}^2$. In the quadratic transformation **E** the point **4** coinciding with **W** corresponds to the point **4'** and thus we concluded that the pencil $\mathbf{p}^2_{1=2=3=4}$ is a pencil of mutually super osculating conics. It has been proved that the theorem which is valid for the case of a set of osculating conics is also valid for the case of the mutually superosculating conics (compare [5], [6] and [7]).

2 Formulation of the problem

If we have the pencil of straight lines **4'** (**a'**, **b'**, **c'**,…) and corresponding to it in the quadratic transformation **E** pencil of conics $\mathbf{p}^2_{1=2=3,4}$ ($\mathbf{a}^2$, $\mathbf{b}^2$, $\mathbf{c}^2$,…) then these two pencils are projective [5]. We can recall also the theorem presented in [6] which states that the pencil of conics $p^2_{1=2=3,4}$, to which conics $\mathbf{a}^2$, $\mathbf{b}^2$, $\mathbf{c}^2$,... belong is in a projective relation to the range of points of the second order with the base on the conic of centers $\mathbf{s}^2$ and the elements $\mathbf{S_a}$, $\mathbf{S_b}$, $\mathbf{S_c}$,... being the centers of the conics $\mathbf{a}^2$, $\mathbf{b}^2$, $\mathbf{c}^2$,... respectively. Based on these two theorems we can formulate the following:

<u>Theorem I.</u> If the pencil of straight lines **4'** (**a'**, **b'**, **c'**,…) is involutory, then the pencil of conics $\mathbf{p}^2_{1=2=3,4}$ ($\mathbf{a}^2$, $\mathbf{b}^2$, $\mathbf{c}^2$,…) or pencil $\mathbf{p}^2_{1=2=3=4}$ ($\mathbf{a}^2$, $\mathbf{b}^2$, $\mathbf{c}^2$,…) corresponding to it in the quadratic transformation **E** is involutory.

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Draft proof. Let a pencil of lines and a pencil of conics corresponding to it in the quadratic transformation **E** be given. From the Theorem ([5], pages 9, 10) it follows that the two corresponding pencils are projective. Analogically, we can conclude the following. If the pencil of straight lines is involutory then the corresponding to it in a quadratic transformation **E** pencil of conics is also involutory.

<u>Theorem II.</u> If any of the pencils $\mathbf{p^2_{1=2=3,4}}$ ($\mathbf{a^2}$, $\mathbf{b^2}$, $\mathbf{c^2}$,…) or $\mathbf{p^2_{1=2=3=4}}$ ($\mathbf{a^2}$, $\mathbf{b^2}$, $\mathbf{c^2}$,…) is involutory then the range of points of the second order to which belong the centers of the conics of the pencil ($\mathbf{p^2}$) is also involutory ([1], pages 182–186).

Draft proof. Let a pencil of conics and the conic of centers $\mathbf{s^2}$ corresponding to it be given in the quadratic transformation **E**. From the Theorem II ([6], page 22) it follows that in the quadratic transformation **E** a pencil of conics is projective to a range of points of the second order for which the base makes the conic of centers $\mathbf{s^2}$. Analogically, we can conclude the following. If the pencil of conics is involutory then the corresponding to it in the quadratic transformation **E** the range of points of the second order whose elements make the conic of centers $\mathbf{s^2}$, is also involutory.

In all the cases discussed here a hyperbolic involution has been considered, as there exist two double real conics in the pencil $\mathbf{p^2}$ ([1], pages 186–188).

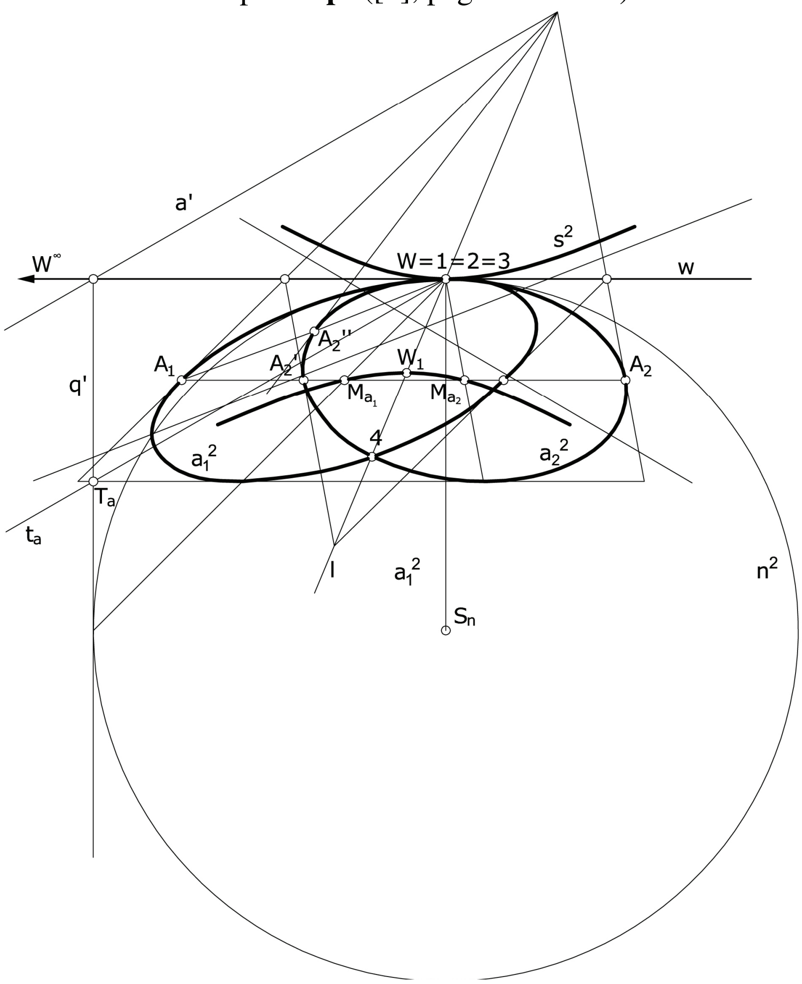

Figure 1a

In Figures 1a and 1b we have drawn the involutory pencils of osculating conics. Let us take the basis $\mathbf{n^2}$, the center **W** of the quadratic transformation **E** and the conic of the centers $\mathbf{s^2}$ of the pencil $\mathbf{p^2_{1=2=3,4}}$. Among the elements of the pencil $\mathbf{p^2_{1=2=3,4}}$ we can distinguish one degenerated conic which is created by two straight lines: the line **w** and the line passing through the points **W** and **4**. Point **W** is the center of this degenerated conic. Let us now assume that the degenerated conic described here is one of the two united elements in the involutory pencil of conics $\mathbf{p}^2_{1=2=3,4}$. The other united element of the pencil is the conic $\mathbf{a}^2$ with the diameter **W4**. The midpoint $\mathbf{W_1}$ of the diameter **W4** is the center of the conic $\mathbf{a}^2$. The points **W** and $\mathbf{W_1}$ of conic $\mathbf{s}^2$ create two double elements in the range of points $\mathbf{s}^2$ while the point at infinity $\mathbf{W}^\infty$ that belongs to the line **w** is both the center of the discussed involution and the pole of the line $\mathbf{W\ W_1}$ (Theorem II). Two corresponding points $\mathbf{Ma_1}$ and $\mathbf{Ma_2}$ which lie on the $\mathbf{s}^2$ are symmetrically positioned in reference to the diameter $\mathbf{W\ W_1}$. A hyperbola (Fig.1a) and an ellipse (Fig.1b) have been taken as a conic of centers $\mathbf{s}^2$ for further considerations. In Figure 1a two corresponding points $\mathbf{Ma_1}$ and $\mathbf{Ma_2}$ have been specified on the conic $\mathbf{s}^2$ as the centers of two conics $\mathbf{a_1}^2$ and $\mathbf{a_2}^2$.

These points make the pair of corresponding elements in the involutory pencil of conics $\mathbf{p}^2_{1=2=3,4}$. The following geometrical relations exist between the conics $\mathbf{a_1}^2$ and $\mathbf{a_2}^2$:

1. Involutory collineation (oblique symmetry) with the center $\mathbf{W}^\infty$ and the axis $\mathbf{l} = \mathbf{W\ W_1}$. In this relation point $\mathbf{A_2}$ corresponds to $\mathbf{A_1}$ while the two tangents at points $\mathbf{A_1}$ and $\mathbf{A_2}$ to the conics intersect on the line **l**;
2. Hirst transformation [8] with the center $\mathbf{W}^\infty$ and the axis **w**. In this relation point $\mathbf{A_2'}$ corresponds to $\mathbf{A_1}$ while the two tangents to the conics at points $\mathbf{A_1}$ and $\mathbf{A_2'}$ meet on the line **w**;
3. Hirst transformation [8] with the center **W** and the axis **l**. In this relation point $\mathbf{A_2''}$ corresponds to point $\mathbf{A_1}$ while the two tangents to the conics at points $\mathbf{A_1}$ and $\mathbf{A_2}$ meet on the line **l** (Fig.1a).

Two corresponding points $\mathbf{Ma_1}$ and $\mathbf{Ma_2}$ have been specified on the conic $\mathbf{s}^2$ (Fig.1b). These points will be the centers of two hyperbolas $\mathbf{a_1}^2$ and $\mathbf{a_2}^2$. Similarly to the case with ellipses (Fig.1a), there exists a perspective relation between the two hyperbolas $\mathbf{a_1}^2$ and $\mathbf{a_2}^2$ as shown in Figure1b.

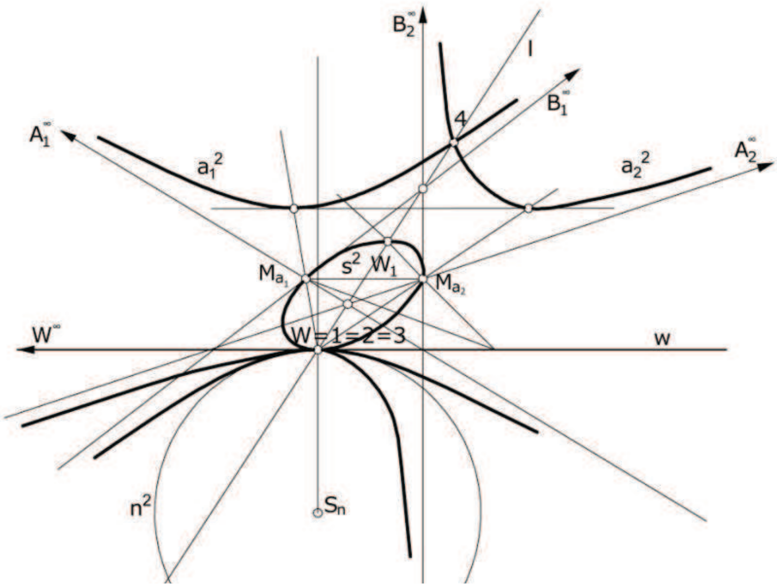

Figure 1b

Figures 2a and 2b show the involutory superosculating pencils of conics. Assume that we have a superosculating pencil of conics $\mathbf{p}^2_{1=2=3=4}$. Let us define the involution of the pencil of conics by taking its two double elements to be: a non–degenerated parabola $\mathbf{p}^2$

and the line **w** that can be considered as a degenerated conic (=a parabola). According to Theorem II, the centers of the conics create an involutory range of points whose elements are: the point $\mathbf{S_p}^{\infty}$ (the center of the parabola $\mathbf{p^2}$) and the point **W** (the center of the degenerated conic). We conclude that the centers $\mathbf{Sa_1}$ and $\mathbf{Sa_2}$ of the conics lie on the line **s** and they are positioned symmetrically with respect to the point **W**.

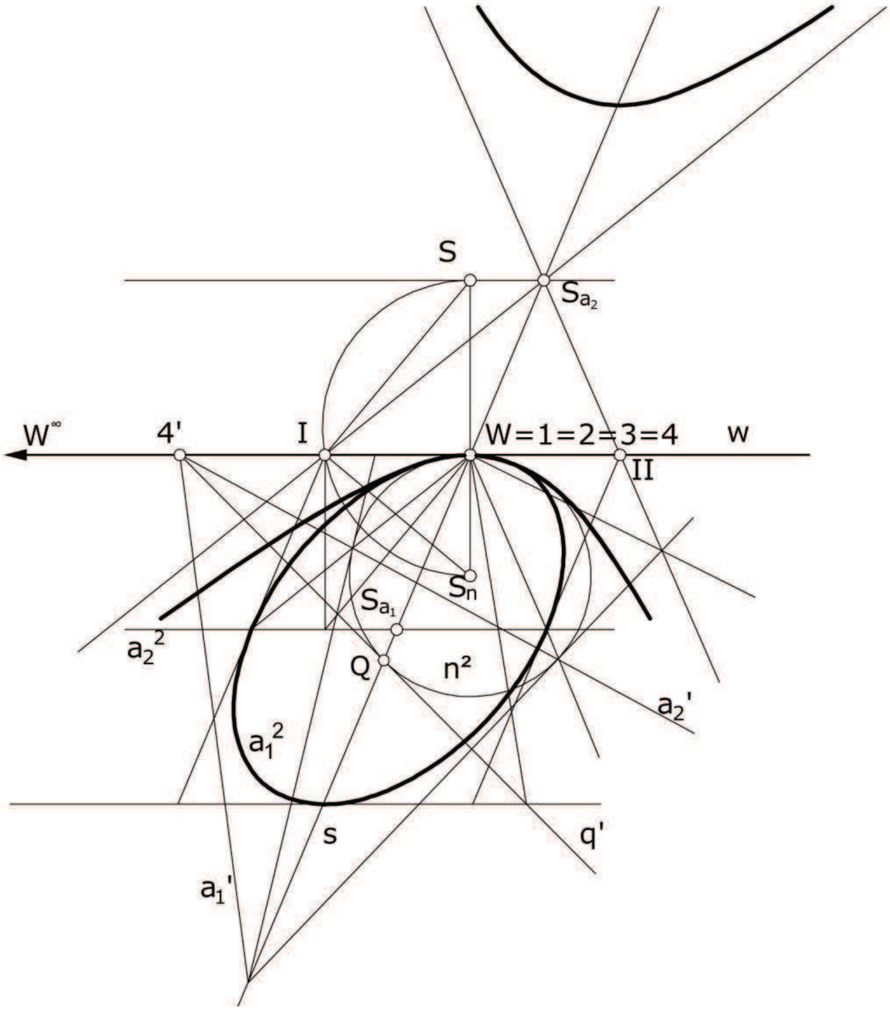

Figure 2a

In Fig 2a. the circle $\mathbf{n^2}$ and the point **W** belonging to the circle have been determined as the basis of the quadratic transformation **E**. We also specify the tangent **w** to the circle $\mathbf{n^2}$ at point **W** and an arbitrary chosen point **4’** on the line **w** that will be a center of the involutory pencil of straight lines **4’** ($\mathbf{a_1}'$, $\mathbf{a_2}'$,…). The involutory pencil of superosculating conics **W** ($\mathbf{a_1^2}$, $\mathbf{a_2^2}$,…) corresponds to the pencil **4’** ($\mathbf{a_1}'$, $\mathbf{a_2}'$,…) (Theorem I). Two tangents **w** and **q’** to the circle $\mathbf{n^2}$ from the point **4’** constitute the double elements in the pencil (**4’**). The secant $\mathbf{a_2}'$ of the circle $\mathbf{n^2}$ corresponds to the arbitrary chosen straight line $\mathbf{a_1}'$ which has an external position in reference to the basic circle $\mathbf{n^2}$. Based on the definition of the quadratic transformation **E** [5] we will determine both the ellipse $\mathbf{a_1^2}$ and the hyperbola $\mathbf{a_2^2}$ corresponding to it. The centers $\mathbf{S_{a1}}$ and $\mathbf{S_{a2}}$ of the conics are positioned symmetrically with respect to the point **W**. In reverse, we can now choose two arbitrary, but lying in a position that is symmetrical with respect to the point **W**, points $\mathbf{S_{a1}}$ and $\mathbf{S_{a2}}$ to be the centers of the respective conics $\mathbf{a_1^2}$ and $\mathbf{a_2^2}$ and then we can determine these conics. Let us note that the tangents to the ellipse $\mathbf{a_1^2}$ meet with the asymptotes of the hyperbola $\mathbf{a_2^2}$ at two points **I** and **II** on the line **w**. If we project the point $\mathbf{S_{a2}}$ from $\mathbf{W}^{\infty}$ to the line $\mathbf{S_nW}$, we get the point **S**. The triangles $\mathbf{S\ I\ S_n}$ and $\mathbf{S\ II\ S_n}$ are right-angled and thus we conclude that the vertices $\mathbf{S\ I\ S_n}$ (and $\mathbf{S\ II\ S_n}$) belong to a semicircle with a diameter $\mathbf{S\ S_n}$ (Fig. 2a)

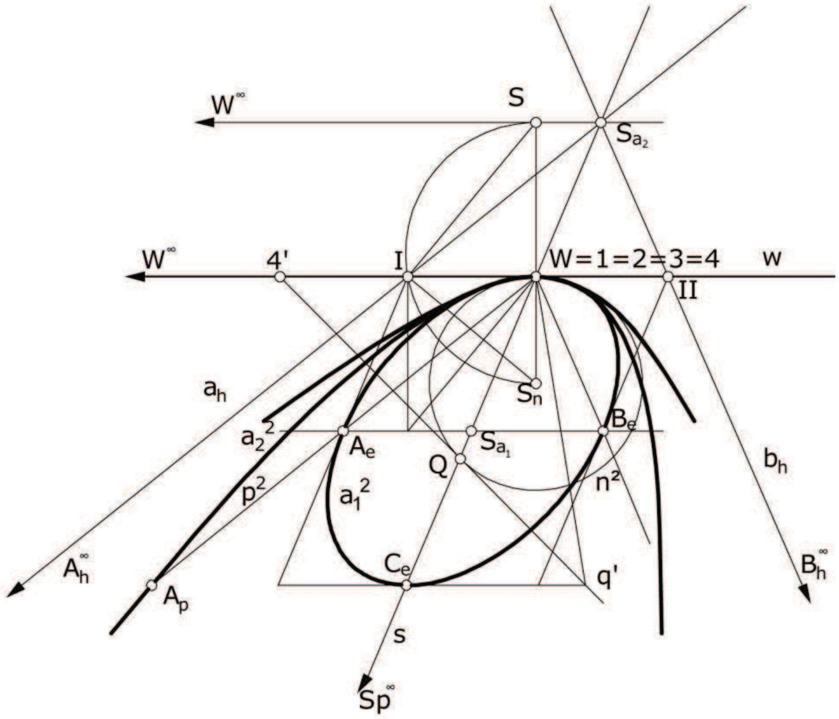

Figure 2b

Let us now take advantage of the properties described above. Let the basic circle $\mathbf{n^2}$, two tangents to the circle $\mathbf{n^2}$ drawn from point **4'**, the tangent **w** at point **W=1=2=3=4** and the tangent **q** to the circle $\mathbf{n^2}$ at point **Q** be given in the transformation **E**. Firstly, two centers $\mathbf{S_{a1}}$ and $\mathbf{S_{a2}}$ of the conics which are to be constructed have been selected [7]. Then we can specify the point **S** (Fig. 2b). The circle with the diameter $\mathbf{S\ S_n}$ intersects the line **w** at two points **I** and **II**. Two lines connecting the point $\mathbf{S_{a2}}$ with **I** and **II** determine two asymptotes of the hyperbola $\mathbf{a_2}^2$ while the line connecting the points $\mathbf{S_{a2}}$ and the center $\mathbf{S_p}^{\infty}$ of the parabola $\mathbf{p^2}$ determine two tangents to the ellipse $\mathbf{a_1}^2$. These two tangents meet the parallel to the line **w** at two points $\mathbf{A_e}$ and $\mathbf{B_e}$. The line $\mathbf{WA_e}$ and the line tangent to the ellipse $\mathbf{a_1}^2$ that is parallel to the line **w** at point $\mathbf{A_p}$ of the parabola $\mathbf{p^2}$ meet at the point $\mathbf{A_p}$ which belongs to the parabola $\mathbf{p^2}$. The segments $\mathbf{A_eW}$, $\mathbf{A_eA_p}$ and **AW** are of equal length. Due to the fact that the conics $\mathbf{w^2}$, $\mathbf{p^2}$, $\mathbf{a_1}^2$ and $\mathbf{a_2}^2$ create a harmonic group ([1] pages 57–62 and 189–191) we conclude that any line passing through the point **W** intersects the set of conics at points which are also harmonic conjugate. The line connecting points **W** and $\mathbf{A_e}$ is parallel to the asymptote $\mathbf{a_h}$ of the hyperbola $\mathbf{a_2}^2$ and thus it intersects with the asymptote at point $\mathbf{A_h}^{\infty}$. If points **W**, $\mathbf{A_p}$, $\mathbf{A_e}$, $\mathbf{A_h}^{\infty}$ are harmonic conjugate, i.e. they create a harmonic set of points ($\mathbf{W\ A_p\ A_e\ A_h}^{\infty}$) = –1, then he segments $\mathbf{W\ A_e}$ and $\mathbf{A_e\ A_p}$ must be of equal length.

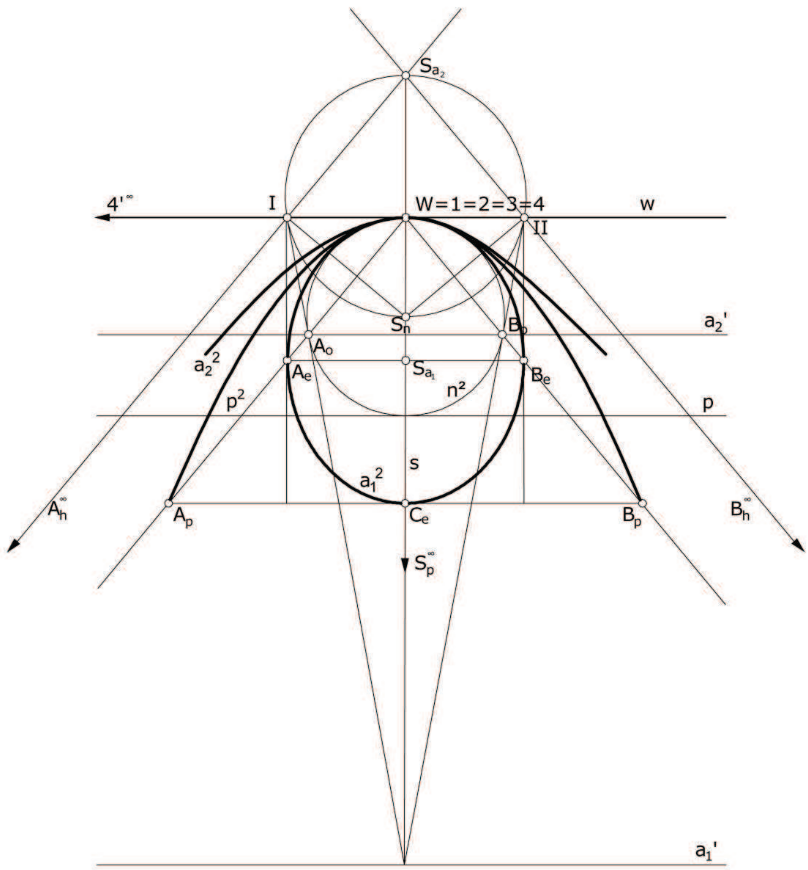

Figure 3

In Figure 3 the tangent **w** is perpendicular to the straight line **s** created by the centers of conics and thus we conclude that the circle $\mathbf{n^2}$ belongs to the pencil $\mathbf{p^2_{1=2=3=4}}$. This case is different from the earlier one described here. The conics $\mathbf{a_1^2}$ and $\mathbf{a_2^2}$ will be obtained in the following way: We determine the circle $\mathbf{n^2}$ as it was done in Figures 2a and 2b with the exception that the lines ($\mathbf{w}$, $\mathbf{p}$, $\mathbf{a_1}'$, $\mathbf{a_2}'$) = –1 are mutually parallel because they are the members of the pencil $\mathbf{4}^{'\infty}$. The circle $\mathbf{n^2}$ will be transformed in the quadratic transformation **E** with the center **W** and the axis **w**. The straight lines **p**, $\mathbf{a_1}$, $\mathbf{a_2}$,... are the vanishing lines of the set to which the circle $\mathbf{n^2}$ belongs ([2] pages 41–45).

Conclusion

Discussion presented in this paper creates the fourth part of the earlier publications which considered the cases of the pencils of osculating and superosculating conics. Two theorems on the involutory pencils of osculating and superosculating conics and the theorem on the involutory range of points of the second order have been shown. The considerations presented here will enhance the ability of solving a wider range of construction problems from the scope of projective geometry than it was possible earlier.

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INWOLUCJE ŚCIŚLE STYCZNYCH PĘKÓW STOŻKOWYCH $\mathbf{P^2_{1=2=3,\,4}}$ ORAZ NADŚCLIŚLE STYCZNYCH PĘKÓW STOŻKOWYCH $\mathbf{P^2_{1=2=3=4}}$

Praca jest kontynuacją artukułów [5, 6, 7]. Przedstawiono w niej dwa twierdzenia; Tw. I: Jeżeli pęk prostych jest **4**' (**a**', **b**', **c**',…) jest pękiem inwolucyjnym, to przyporządkowany mu w przekształceniu kwadratowym **E** pęk stożkowych ściśle lub nadściśle stycznych jest również pękiem inwolucyjnym. Tw. II: Jeżeli pęk stożkowych ściśle lub nadściśle stycznych jest pękiem inwolucyjnym, to szereg punktów rzędu drugiego, którego elementami są środki stożkowych pęku ($\mathbf{p^2}$) jest również szeregiem inwolucyjnym.