PENCILS OF THE MAUTUALLY SUPER OSCULATING CONICS $P^2_{1=2=3=4}$

Barbara WOJTOWICZ[1], Ada PAŁKA[2]

[1] Cracow University of Technology
Division of Descriptive Geometry, Technical Drawing and Engineering Graphics
Warszawska st. 24, 31-155 Kraków, Poland
email: wojtowiczbm@gmail.com
[2)] Jagiellonian University
Faculty of Mathematics & Computer Science
Prof. S. Łojasiewicz st. 6, 30-348 Kraków, Poland
email: ada.palka@uj.edu.pl

Abstract. The E- transformation is a quadratic transformation in the projective 2D space for which the base constitute the circle n^2 and the center W which lies on this circle. Specifically, the authors present the results of the further discussion on the properties of the pencils of super osculating conics. The theorem on projective relation between the elements of the pencil of super osculating conics and the range (of the second order) of the conics' centers has been proved.

Keywords: projective geometry, pencils of conics, square transformation E, elation

1 Introduction

In this paper the author presents the results of the research work which is a continuation of the earlier considerations that have been published in [4] and [5]. The quadratic transformation **E** has been defined in the following way: the circle $\mathbf{n^2}$ is its base and the point **W** belonging to the base circle $\mathbf{n^2}$ has been chosen to be the center of the **E** transformation. All lines passing through the point **W** will be transformed into a set of osculating conics that pass through the point **1=2=3=W**. The centers of the osculating conics create the conic $\mathbf{s^2}$, which has been called the conic of the centers.

2 Formulation of the problem

Let the base $\mathbf{n^2}$ of the transformation **E** be given together with a pencil of straight lines. Let the vertex **4'** of this pencil belong to the line **w** (Fig.1). In the **E** transformation the corresponding elements are as follows:

- line **q'** of the pencil **(4')** corresponds to a parabola $\mathbf{q^2}$,
- line **a'** which does not intersect the base circle $\mathbf{n^2}$ corresponds to the ellipse $\mathbf{a^2}$,
- the secant **b'** of the base circle $\mathbf{n^2}$ corresponds to the hyperbola $\mathbf{b^2}$.

In the discussed layout the conic of centers $\mathbf{s^2}$ will become degenerated into two straight lines **q** and **w**. The last can be symbolically denoted as $\mathbf{s^2 = q + w}$. All the vertices of the basic quadrangle **1=2=3=4** become united at a single point **W** and thus the conic $\mathbf{s^2}$ degenerates into two lines: $\mathbf{q=WQ^0}$ and **w = W4'** (compare [1], pp. 146, 147). Point **4** that coincides with the point **W** corresponds to point **4** and thus the pencil of conics $\mathbf{p^2_{1=2=3=4}}$ creates a pencil of super osculating conics. The pencil of conics $\mathbf{p^2_{1=2=3=4}}$ consists of the following elements:

- the set of **ellipses** corresponding to the lines of the pencil (4') which are in the "external" position in reference to the base circle $\mathbf{n^2}$,

- the set of **hyperbolas** corresponding to the secants of the base circle $\mathbf{n^2}$,
- two **parabolas**; one of them is a non-degenerated parabola $\mathbf{q^2}$, which corresponds to the tangent line **q** and the other one is a **parabola** which has been degenerated to the line **w**.

Whereas the line **q** is the polar line of the point **4'** in reference to the circle $\mathbf{n^2}$, and whereas the center $\mathbf{M_n}$ of the circle $\mathbf{n^2}$ does not belong to the line **q**, we can conclude that the circle $\mathbf{n^2}$ does not belong to the pencil $\mathbf{p^2_{1=2=3=4}}$ (Fig.1).

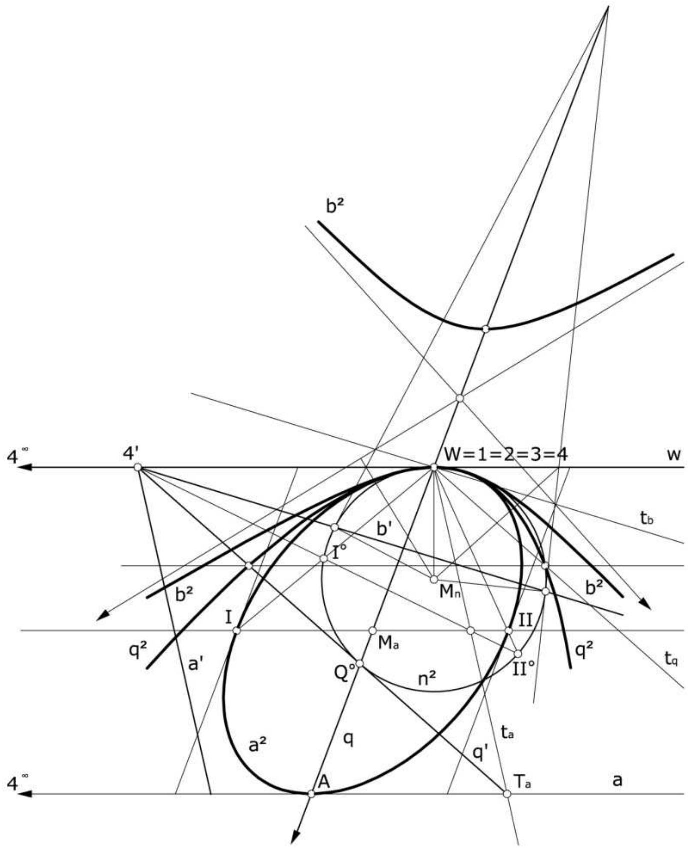

Figure1

Thesis: We will now prove that for the case of a pencil of super osculating conics $\mathbf{p^2_{1=2=3=4}}$ the theorem considering osculating conics is also true.

Theorem

The **pencil** of conics $\mathbf{p^2_{1=2=3=4}}$ which consists of the conics $\mathbf{a^2}$, $\mathbf{b^2}$, $\mathbf{q^2}$,… is in **projective relation** to the **range of the second order** with the base on the conic $\mathbf{s^2}$ and the points $\mathbf{M_a}$, $\mathbf{M_b}$, $\mathbf{M_q}$,…. becoming the elements of this range, while these points are the centers of the corresponding conics $\mathbf{a^2}$, $\mathbf{b^2}$, $\mathbf{q^2}$, …

Proof: In order to construct the center $\mathbf{M_a}$ of the conic $\mathbf{a^2}$ which corresponds to the line **a'** of the pencil **(4')** we need to perform the subsequent steps of construction:

- Let us draw a straight line $\mathbf{t_a}$ parallel to the line **a'** and passing through the point **W** (line $\mathbf{t_a}$ is the axis of the elation in the quadratic transformation **E** [4, pp.5, 6],
- Lines $\mathbf{t_a}$ and **q'** meet at point $\mathbf{T_a}$,
- Through the point $\mathbf{T_a}$ let us draw the line **a** parallel to the line **w**. Line **a** will be the other tangent line to the conic $\mathbf{a^2}$ (an ellipse, as the **a'** is the external secant of the base circle $\mathbf{n^2}$,
- Lines **a** and **q** (line **q** is the **polar** line of point **4'** in relation to the circle $\mathbf{n^2}$) meet in point **A** which is the other vertex of the conic's axis,

- The center $\mathbf{M_a}$ of the conic $\mathbf{a^2}$ is a midpoint of the segment **AW**, which is the axis of the conic.

If we continue performing the above described activity for the subsequent lines of the pencil **(4')**, we will obtain the **perspective range** of **the sets of the 1st degree elements** (**pencils of lines** and **ranges of points**) as described below:

$$\mathbf{4'\ (a', b', q', \ldots) \overset{pers}{\leftrightarrow} n'^{\infty}\ (A'^{\infty}, B'^{\infty}, Q'^{\infty}, \ldots) \overset{pers}{\leftrightarrow} W\ (t_a, t_b, t_q, \ldots) \overset{pers}{\leftrightarrow} q'\ (T_a, T_b, T_q, \ldots)}$$

$$\overset{pers}{\leftrightarrow}$$

$$\mathbf{4^{\infty}\ (a, b, q, \ldots) \overset{pers}{\leftrightarrow} q\ (A, B, Q, \ldots)}$$

The boundary elements of this range are **projective**, e.a.

$$\mathbf{4'\ (a', b', q', \ldots) \overset{proj}{\leftrightarrow} q\ (A, B, Q, \ldots\ldots),}$$

where the symbols:

$\overset{proj}{\leftrightarrow}$ and $\overset{pers}{\leftrightarrow}$ will be used to denote the projectivity and the perspectivity.

As points $\mathbf{M_a}$, $\mathbf{M_b}$, $\mathbf{M_q}$ are the midpoints of the relevant segments **WA, WB, WQ,…** thus the range **q (A**, **B**, **Q, …...)** is in perspective relation to the range **q (**$\mathbf{M_a}$, $\mathbf{M_b}$, $\mathbf{M_q}$**, …...)**, and hence we can conclude that the ranges
4' (a', b', q',…) and q (M_a, M_b, M_q, …...) are projective (as stated).

From the fact that the base circle $\mathbf{n^2}$does not belong to a discussed pencil $\mathbf{p^2_{1=2=3=4}}$, we can conclude that the vertices of a basic **quadrangle** must not be chosen to lie on the circle $\mathbf{n^2}$.

To give an example the vertices can be arbitrarily chosen on a parabola (Fig.2).

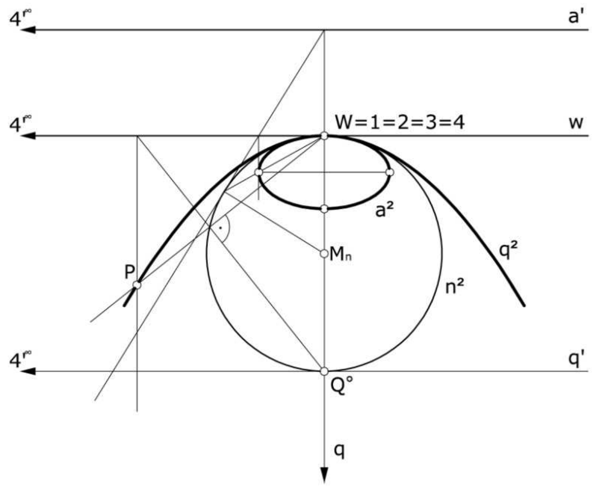

Figure 2

In Fig.2 an arbitrary parabola $\mathbf{q^2}$ has been given. The following elements of the parabola have been chosen: its diameter **q**, a conjugate to this diameter tangent **w** and an optional point **P**. The pencil of conics $\mathbf{p^2_{1=2=3=4}}$ will be determined when the vertices of a basic quadrangle **1=2=3=4** coincide with the point **W=qw**. Let us determine the osculating circle $\mathbf{n^2}$ to the parabola $\mathbf{q^2}$ at point **W**. Line **q** meets the circle $\mathbf{n^2}$ at point $\mathbf{Q^0}$. The tangent **q'** to the circle $\mathbf{n^2}$ at point $\mathbf{Q^0}$ meets the line **w** at point **4'.** The last point is the vertex of a pencil of lines which will be transformed in the **E** into a pencil of mutually over-osculating circles $\mathbf{p^2_{1=2=3=4}}$.

Let the circle $\mathbf{n^2}$ be optionally chosen. If we arbitrarily choose the coinciding points **1=2=3=4** on the circle $\mathbf{n^2}$ so that they unite with the point **W** then we obtain a pencil of

ISSN 1644-9363 / PLN 15.00 © 2012 PTGiGI

mutually super osculating conics, while the circle $\mathbf{n^2}$ belongs to the pencil (Fig.3). The point at infinity $\mathbf{4'}^{\infty}$ belonging to the line **w** will become the pencil's vertex. This pencil will get transformed into a pencil of conics. Line **w** is the axis of elation for the **E** transformation. In Fig.3 two specific lines of the pencil $\mathbf{4'}^{\infty}$ have been distinguished. One of them is the line **q'** which is the tangent to the circle $\mathbf{n^2}$ and the other is, the external of the circle $\mathbf{n^2}$, the line **a'**. Two conics: a parabola $\mathbf{q^2}$ and an ellipse $\mathbf{a^2}$ respectively correspond to the distinguished lines.

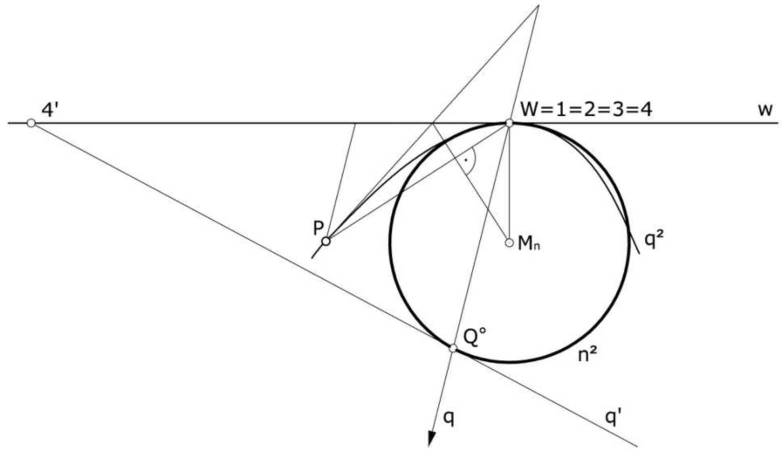

Figure 3

3 Conclusions

Discussion presented in this paper together with the earlier considerations will aid solving various construction problems from the scope of descriptive geometry.

References

[1] Plamitzer A.: *Elementy geometrii rzutowej*. Politechnika Lwowska, Lwów, 1927.
[2] Hilbert D.,Cohn-Vossen S.: *Geometria poglądowa*. PWN, Warszawa, 1956.
[3] Szerszeń S.: *Nauka o rzutach*. PWN, Warszawa, 1974.
[4] Wojtowicz B.: *Pencil of osculary tangent conics $p^2_{1=2=3=4}$*. The Journal *Biuletyn* of Polish Society for Geometry and Engineering Graphics, Vol. 17, Gliwice, 2007.
[5] Wojtowicz B.: *The conic of centers s^2 of a pencil $p^2_{1=2=3=4}$*. The Journal *Biuletyn* of Polish Society for Geometry and Engineering Graphics, Vol.18, Gliwice, 2008.

PĘKI STOŻKOWYCH WZAJEMNIE NADŚCIŚLE STYCZNYCH $\mathbf{P^2_{1=2=3=4}}$

Praca jest kontynuacją artykułu [4]: *Pęki stożkowych ściśle stycznych $p^2_{1=2=3=4}$* oraz artykułu [5]: *Stożkowe środków pęku $p^2_{1=2=3=4}$*, w których omówiono przekształcenie kwadratowe E. Bazą przekształcenia jest okrąg n^2, a środkiem przekształcenia punkt W leżący na tym okręgu. Stwierdzono, iż wszystkie proste , które przechodzą przez punkt W przekształcają się w stożkowe wzajemnie ściśle styczne przechodzące przez trzy punkty 1=2=3 pokrywające się z punktem W. Środki poszczególnych stożkowych pęku leżą na stożkowej, którą nazwano stożkową środków i oznaczono s^2. W pracy udowodniono twierdzenie o relacji rzutowej między elementami pęku stożkowych nadściśle stycznych a szeregiem drugiego rzędu, którego elementami są środki stożkowych, które powstają w wyniku zastosowania transformacji E.