USING A CAS TO VISUALIZE SOME IMAGES OF LINES MAPPED VIA THE HARMONIC CROSS-RATIO

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Abstract. Let K be a cubic curve in the projective space $\boldsymbol{P}^3$ and let T_1 and T_2 be points determining a bisecant T_1T_2 of K. We fix a point A on K and a point $B \neq A$ which does not lay on K, and such that $T_1T_2 \neq AB$. We are interested in the set of points X generated by the equation $(T_1, T_1; M, X) = -1$ where M denotes the point at which AB meets the bisecant T_1T_2. So we consider the line congruence of order 1 and of class 3 in the aspect of the harmonic cross-ratio. We derive theoretic formulas for the set of X 's and we go on in the harmonic case – then the set of X 's is a conic. We use the computer algebra system Derive 5 from Texas Instruments, Inc., USA, to produce visualizations of the images of resulting curves.

Keywords: projective geometry, cross-ratio, conics, bisecant, visualization, computer algebra system

1 Basic notions on three-dimensional projective space

As usually, $\boldsymbol{P}^3$ denotes a real projective space of dimension 3, a quadruple $b = (b_1 : b_2 : b_3 : b_4)$ is the natural identification of a (**regular**, **usual**) **point** in $\boldsymbol{P}^3$ if $b_4 \neq 0$, and it is said to be a **point in the infinity** if $b_4 = 0$. Numbers b_1, b_2, b_3, b_4 are called **homogeneous coordinates** of b. In the Cartesian space $\boldsymbol{R}^3$ the **natural representation** of a point $b = (b_1 : b_2 : b_3 : b_4)$ is the point $(b_1/b_4,\ b_2/b_4,\ b_3/b_4)$, and a point $(b_1 : b_2 : b_3 : 0)$, called a **direction**, can be seen as a direction of the vector $[b_1, b_2, b_3]^T$. For any $b, c \in \boldsymbol{P}^3$ we write $b \equiv c$, and we say b and c are **equivalent**, iff there exists a nonzero number λ (called a **homogeneity multiplier**) such that $b = \lambda \cdot c$.

Any curve in $\boldsymbol{P}^3$ can be described by the set of four equations in one variable. The formula $p(t) = (1 : t : t^2 : t^3)$, where t runs from $-\infty$ to $+\infty$, defines the spatial curve in $\boldsymbol{P}^3$. In appropriately chosen coordinate system this formula describes the **cubic curve**; it is an example of a skew curve, we denote it by K and we call it a **standard**, or **reference**, **curve**.

A **line congruence** in $\boldsymbol{P}^3$, or a **congruence of lines**, is defined as a two-parameter family of lines in $\boldsymbol{P}^3$. Here we deal with the (1, 3)-congruence in $\boldsymbol{P}^3$, i.e., the line congruence of order 1 and of class 3. In the terminology used in Grassmann varieties, we deal with the set of all 1-dimensional subspaces of a projective space of dimension 3. It says that

1) there is exactly one line congruence that passes through an arbitrary point of $\boldsymbol{P}^3$,
2) in any plane $\boldsymbol{P}^2$ there are exactly 3 lines belonging to the line congruence at hand.

The study of line congruences has been very popular in the turn of two last centuries and it is still investigated all over the world (see, e.g. [1], [2], [4]-[7]).

A line which cuts the given curve at exactly two points is called a **bisecant** of this curve. It was proved by Ernst Kummer in 1866 (and reproved in 1986 by Ziv Ran) that there

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are exactly two different types of (1,3)-congruences: 1) the congruence of bisecants to a twisted cubic, 2) the congruence of lines meeting a twisted cubic and a bisecant to it, or a degeneration of it. We consider the first of these cases.

Let's take two distinct values t_1, t_2. They produce two distinct points $T_1 := K(t_1)$, $T_2 := K(t_2)$ sitting on the curve K. It is well-known (see, e.g. [3]) that the straight line T_1T_2, i.e., the line passing through points T_1 and T_2 is the bisecant of our twisted curve K.

2 Cross-ratio of four numbers and four points

As usual, a **cross-ratio**, or **double ratio**, determined by distinct numbers a, b, c, d is

$$(a, b; c, d) := \frac{a-c}{a-d} \cdot \frac{b-d}{b-c}.$$

Usually its value is denoted by λ. In the next we investigate in details the case $\lambda = -1$. Then we have so-called **harmonic division** and we deal with a **harmonic cross-ratio**.

A **cross-ratio** of four points, A, B, C, D, is defined by the same formula as that of four numbers, but now a, b, c and d stand for the number identifiers of these points in the local coordinate system; usually, $a - b$ is the signed distance between points A and B. As in the number case, the cross-ratio of points is denoted by $(A, B; C, D)$.

3 Coordinates of the point M

Let's take a point $A = K(a) = (1 : a : a^2 : a^3)$ and a point $B = (b_1 : b_2 : b_3 : b_4) \notin K$ such that the line AB is not a bisecant of K. Next, let's take an arbitrary point M on AB. We discuss the line congruence of order 1, so there exists [3] exactly one bisecant of K passing through M. Let's denote the points, at which this bisecant intersects K, by $T_1 := K(t_1)$, $T_2 := K(t_2)$.

Since the point $M = (m_1 : m_2 : m_3 : m_4) \in T_1T_2$, so there exist reals α, β such that

$$M \equiv \alpha \cdot T_1 + \beta \cdot T_2. \tag{1}$$

Since M is different from both T_1 and T_2, so $\alpha^2 + \beta^2 > 0$. Therefore the equation (1) is

$$\begin{cases} \rho \cdot m_1 = \alpha + \beta \\ \rho \cdot m_2 = \alpha \cdot t_1 + \beta \cdot t_2 \\ \rho \cdot m_3 = \alpha \cdot t_1^{\,2} + \beta \cdot t_2^{\,2} \\ \rho \cdot m_4 = \alpha \cdot t_1^{\,3} + \beta \cdot t_2^{\,3} \end{cases}. \tag{2}$$

where ρ is a homogeneity multiplier. From two first equations of the system (2) we obtain

$$\alpha = \frac{m_1 \cdot t_2 - m_2}{t_2 - t_1} \cdot \rho, \beta = \frac{m_2 - m_1 \cdot t_1}{t_2 - t_1} \cdot \rho. \tag{3}$$

It shows than we can work on with $\rho = 1$ and we have

$$s = \frac{m_1 \cdot m_4 - m_2 \cdot m_3}{m_1 \cdot m_3 - m_2^{\,2}}, \; p = \frac{m_2 \cdot m_4 - m_3^{\,2}}{m_1 \cdot m_3 - m_2^{\,2}} \tag{4}$$

where

$$s := t_1 + t_2, \; p := t_1 \cdot t_2. \tag{5}$$

In consequence, unambiguously to the numeration, we get

$$t_1 = \frac{s + \sqrt{\Delta_1}}{2}, \; t_2 = \frac{s - \sqrt{\Delta_1}}{2}, \tag{6}$$

where

$$\Delta_1 := s^2 - 4p = \frac{(m_1 \cdot m_4 - m_2 \cdot m_3)^2 - 4 \cdot (m_2 \cdot m_4 - m_2^{\,2}) \cdot (m_1 \cdot m_3 - m_2^{\,2})}{(m_1 \cdot m_3 - m_2^{\,2})^2}. \tag{7}$$

Values t_1 and t_2 produce points T_1 and T_2 laying on K. Substituting (5) in (3) we get the values of coefficients α, β and this way we have the point $M \equiv \alpha \cdot T_1 + \beta \cdot T_2$.

4 Finding the conjugate to given point to have the fixed cross-ratio

The bisecant T_1T_2 and the straight line $AB \neq T_1T_2$ meet at M. Now we look for a point X which lays on T_1T_2 and completes the triple (T_1, T_2, M) in such a way that points T_1, T_2, M and X form their cross-ratio equal to a given real λ. This four can be arbitrarily ordered, so there can be two such points. First, we look for $X = (x_1 : x_2 : x_3 : x_4)$ such that

$$(T_1, T_2; M, X) = \lambda. \tag{8}$$

Taking into account that M lays on the line AB, the demand (8) gives that

$$\begin{cases} \rho \cdot x_1 = \lambda \cdot \alpha + \beta \\ \rho \cdot x_2 = \lambda \cdot \alpha \cdot t_1 + \beta \cdot t_2 \\ \rho \cdot x_3 = \lambda \cdot \alpha \cdot t_1^{\,2} + \beta \cdot t_2^{\,2} \\ \rho \cdot x_4 = \lambda \cdot \alpha \cdot t_1^{\,3} + \beta \cdot t_2^{\,3}, \end{cases} \tag{9}$$

where ρ is an homogeneity multiplier, and α, β are coefficients given by (3).
Taking into account the formulas (3) and (5) we derive following representation

$$\begin{cases} \rho \cdot x_1 = (\lambda - 1) \cdot \psi_1 - (\lambda + 1) \cdot m_1 \cdot \sqrt{\Delta_2} \\ \rho \cdot x_2 = (\lambda - 1) \cdot \psi_2 - (\lambda + 1) \cdot m_2 \cdot \sqrt{\Delta_2} \\ \rho \cdot x_3 = (\lambda - 1) \cdot \psi_3 - (\lambda + 1) \cdot m_3 \cdot \sqrt{\Delta_2} \\ \rho \cdot x_4 = (\lambda - 1) \cdot \psi_4 - (\lambda + 1) \cdot m_4 \cdot \sqrt{\Delta_2} \end{cases}, \tag{10}$$

where $\psi_1 := m_1 \cdot m_1 \cdot m_3 - 3m_1 \cdot m_2 \cdot m_3 + 2m_2 \cdot m_2 \cdot m_2$,
$\psi_2 := m_1 \cdot m_2 \cdot m_4 - 2m_2 \cdot m_3 \cdot m_3 + m_2 \cdot m_2 \cdot m_3$,
$\psi_3 := 2m_2 \cdot m_2 \cdot m_3 - m_1 \cdot m_3 \cdot m_4 - m_2 \cdot m_3 \cdot m_3$,
$\psi_4 := 3m_2 \cdot m_3 \cdot m_4 - 2m_3 \cdot m_3 \cdot m_3 - m_1 \cdot m_4 \cdot m_4$,
$\Delta_2 := (m_1 \cdot m_4 - m_2 \cdot m_3)^2 - 4 \cdot (m_2 \cdot m_4 - m_3 \cdot m_3) \cdot (m_1 \cdot m_3 - m_2 \cdot m_2)$.

Therefore the above representation writes down as follows

$$\rho \cdot x = (\lambda - 1) \cdot \psi - (\lambda + 1) \cdot m \cdot \sqrt{\Delta_2}, \tag{11}$$

where x, ψ and m stand for the vectors with components x_j, ψ_j and m_j, respectively.

Now we look for the point Y such that

$$(T_1, T_2; Y, M) = 1/\lambda. \tag{12}$$

Proceeding analogously as above, we derive the representation

$$\rho \cdot y = (\lambda - 1) \cdot \psi + (\lambda + 1) \cdot m \cdot \sqrt{\Delta_1}. \tag{13}$$

We see that the representations of both points, x and y, differ only in the way in which the second term is involved: in (11) it is subtracted, while in (13) it is added. For the harmonic cross-ratio, i.e., when $\lambda = -1$, the second term gives no share and $x = y$.

Since now we continue our research with $\lambda = -1$. Then $1/\lambda = \lambda$ and we have only one point $X = (x_1 : x_2 : x_3 : x_4)$ such that

$$(T_1, T_2; M, X) = -1. \tag{14}$$

Taking into account the parametric representation,

$$M = A \cdot u + B,$$

of the line AB and combining all above we express the coordinates of the point M in terms of the coordinates of A, B and both T_1, T_2. This way we derive the relation

$$\rho \cdot x = -2 \cdot \psi, \tag{15}$$

where $\psi_j = \psi_j(u)$ are polynomials (of second degree in the variable u) given by formulas

$\psi_1 := \{b_4 - 3a{\cdot}b_3 + 3a^2{\cdot}b_2 - a^3{\cdot}b_1)\}{\cdot}u^2 + \{2b_1{\cdot}b_4 + a^3{\cdot}b_1^2 - 3a{\cdot}b_1{\cdot}b_3 - 3b_2{\cdot}b_3 + 6a{\cdot}b_1^2 - 3a^2{\cdot}b_1{\cdot}b_2\}{\cdot}u + b_1{\cdot}(b_1{\cdot}b_4 - 3b_2{\cdot}b_3) + 2b_2^3,$

$\psi_2 := a{\cdot}\{b_4 - 3a{\cdot}b_3 + 3a^2{\cdot}b_2 - a^3{\cdot}b_1\}{\cdot}u^2 + \{2b_3{\cdot}(a{\cdot}b_2 - b_3) + (a{\cdot}b_1 + b_2){\cdot}(a^2{\cdot}b_2 + b_4) - 4a^2{\cdot}b_1{\cdot}b_3\}{\cdot}u + b_2{\cdot}(b_1{\cdot}b_4 + b_2{\cdot}b_3) + 2\, b_1{\cdot}b_3^2,$

$\psi_3 := a^2{\cdot}\{b_4 - 3a{\cdot}b_3 + 3a^2{\cdot}b_2 - a^3{\cdot}b_1\}{\cdot}u^2 + \{2a^2{\cdot}b_2{\cdot}(a{\cdot}b_2 - b_3) - (a^2{\cdot}b_1 + b_3){\cdot}(a{\cdot}b_3 + b_4) + 4a{\cdot}b_2{\cdot}b_4\}{\cdot}u + 2b_2^2{\cdot}b_4 - b_3{\cdot}(b_1{\cdot}b_4 + b_2{\cdot}b_3),$

$\psi_4 := a^3{\cdot}(b_4 - 6a{\cdot}b_4 + 3a^2{\cdot}b_2 + 3a{\cdot}b_3 - a^3{\cdot}b_1){\cdot}u^2 + \{3a{\cdot}b_4{\cdot}(a{\cdot}b_2 + b_3 - 2a{\cdot}b_4) - b_4{\cdot}(b_4 + 2a^3{\cdot}b_1) + 3a^3{\cdot}b_2{\cdot}b_3\}{\cdot}u + b_3{\cdot}(3b_2{\cdot}b_4 - b_1{\cdot}b_3) - 2b_4^3,$

The curve governed by (15) will be referred to as the curve (a, b)-adjoint to the line K, or, shortly, **(a, b)-curve**. The same terminology will be used for the natural representation of this curve in the space $\boldsymbol{R}^3$ embedded in the standard Cartesian coordinate system $Oxyz$, i.e., it will be used for the line described parametrically as follows

$$(x, y, z) = \left(\frac{\psi_1(u)}{\psi_4(u)}, \frac{\psi_2(u)}{\psi_4(u)}, \frac{\psi_3(u)}{\psi_4(u)}\right). \qquad (16)$$

5 Some remarks on the image of a straight line in $\boldsymbol{P}^3$

Formula (15) reveals that the set of all points X generated by M running the line AB is an algebraic curve in $\boldsymbol{P}^3$ of the 2nd degree. In consequence, its natural representation (16) in $\boldsymbol{R}^3$ determines an algebraic curve, and each component in (16) is a rational function of the degree at most 2 in both nominator and denominator.

We undertake the task to recognize this representation. We will do it via the analysis of its projection on basic planes, i.e., Oxy, Oxz and Oyz planes. Below we report examples with $a = 1$ (it does not affect the generality of the considerations) and $b_4 = 1$ or $b_4 = 3$, and we conclude that all produced curves are plane algebraic curves of degree 2, so each one curve is a hyperbola, a parabola or an ellipse. The type of the curve depends on the expression for the function ψ_4 as follows: a) if $\psi_4(u) > 0$ for all u, then we have an ellipse,

b) if ψ_4 has one zero, then we have a parabola,

c) if $\psi_4(u) = 0$ for two distinct values of u, then we have a hyperbola.

If $a = b_4 = 1$, then the task aiming to recognize the type of (a, b)-curve is dramatically simple: we have three parameters to be varied and the equation $\psi_4(u) = 0$ turns into the system equations (in unknown b_1, b_2, b_3):

$-5 + 3b_2 + 3b_3 - b_1 = 0,\ 3b_2 + 3b_3 - 7 + 2b_1 + 3b_2{\cdot}b_3 = 0,\ b_3{\cdot}(3b_2 - b_1{\cdot}b_3) - 2 = 0.$

We can use a computer algebra system (CAS), e.g., Derive 5 for Windows, to obtain solutions, (b_1, b_2, b_3), they are

(0, 1, 2/3), (1, 1, 1), (–5, 1, –1), (–1/3, 5/9, 1) and (α, (α+2)/3, 1),

where α is an arbitrary real. One of these solutions, namely (1, 1, 1), identifies the point (1 : 1: 1 : 1), so we throw it out. The value $\alpha = -1/3$ yields the fourth of the triples. Therefore all solutions we have are

$$(0, 1, 2/3), (-5, 1, -1), (\alpha, (\alpha+2)/3, 1). \qquad (17)$$

The situation remains similar if b_4 is an arbitrary non-zero number. For instance, when $b_4 = 3$, then the equation $\psi_4(u) = 0$ yields the system:

$-15 + 3b_2 + 3b_3 - b_1 = 0,\ -21 + 3b_2 + 3b_3 + 2b_1 + b_2{\cdot}b_3 = 0,\ b_3{\cdot}(9b_2 - b_1{\cdot}b_3) - 54 = 0.$

Its solutions, (b_1, b_2, b_3), are

(0, 3, 2), (–1, 5/3, 3), (3, 3, 3), (–15, 3, –3) and (β, (β+6)/3, 3),

where β is an arbitrary real. Eliminating, as above, the non-acceptable triple (3, 3, 3) and the doubled point (–1, 5/3, 3) we finally have

$$(0, 3, 2), (-15, 3, -3), (\beta, (\beta+6)/3, 3). \qquad (18)$$

Obviously, the solutions (17) and (18) are identical in $\boldsymbol{P}^3$, namely they determine the points (0 : 3 : 2 : 1), (–5 : 1 : –1 : 1) and, substituting $\beta = 3\alpha$, the point $(\alpha : (\alpha+2)/3 : 1 : 1)$.

6 Examples of the visualization

As it was said, below we have $\lambda = -1$ and $a = 1$, and we give some examples of resulting conics. In some examples we get the equations of planes in which these conics lay.

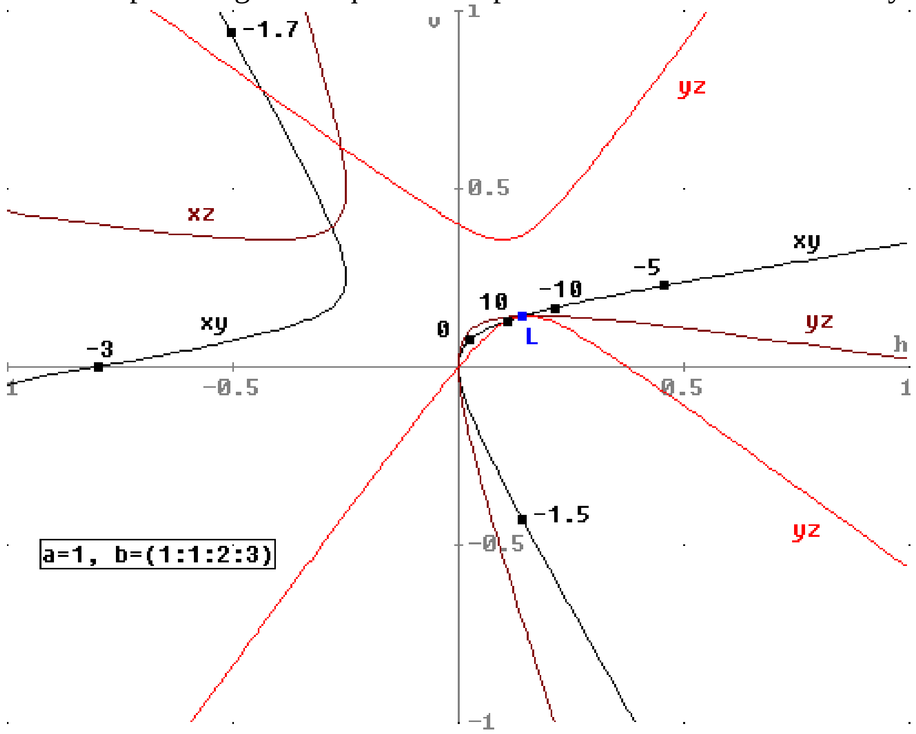

Figure 6.1: Curves *xy*, *xz* and *yz*, traced in the standard Cartesian system *Ohv* and governed by the equations $(h, v) = (x(u), y(u))$, $(h, v) = (x(u), z(u))$ and $(h, v) = (y(u), z(u))$, respectively, when $a = 1$, $b = (1: 1 : 2 : 3)$. On the hyperbola *xy* there are marked the points produced for chosen values of the parameter u sitting in the intervals $(-\infty, z_1)$, (z_1, z_2) and $(z_2, +\infty)$, namely $u = -10$ and -5, $u = -3$ and -1.7, $u = -1.5$, 0 and 10. $L = (1/7, 1/7)$ stands for the limiting point, i.e., the point produced when $u \to z_1 \approx -3.51903$

6.1 First hyperbolic example

For $b = (1 : 1 : 2 : 3)$ we have $\psi_1(u) = u^2 + 2u + 1$, $\psi_2(u) = u^2 + 4u + 3$,
$(u) = u^2 + 5u + 4$, $\psi_4(u) = 7u^2 + 36u + 40$.

It results with the (a, b)-curve in $\boldsymbol{R}^3$. In *Oxyz* system this curve is governed by the equation

$$x(u) = \frac{u^2+2u+1}{7u^2+36u+40},\ y(u) = \frac{u^2+4u+3}{7u^2+36u+40},\ z(u) = \frac{u^2+5u+4}{7u^2+36u+40},$$

where $u \in (-\infty, z_1) \cup (z_1, z_2) \cup (z_2, +\infty)$, z_1 and z_2 are zeros of the equation $\psi_4(u) = 0$, i.e.,

$$z_1 = \frac{-18-2\sqrt{11}}{7} \approx -3.51903,\ z_1 = \frac{-18+2\sqrt{11}}{7} \approx -1.62382.$$

It is easy to notice (see Fig.6.1) that

a. the $(1, b)$-curve passes through the origin O (for $u = -1$),
b. it approaches the point (1/7, 1/7, 1/7) when u tends to both $-\infty$ and $+\infty$,
c. the examination of the equation

$$A\cdot\psi_1(u) + B\cdot\psi_2(u) + C\cdot\psi_3(u) = D\cdot\psi_4(u) \qquad (19)$$

gives that it can takes place only for $D = 0$; then $A = 1$, $B = -3$, $C = 2$ and, consequently, it reveals that the curve $(x, y, z) = (\,x(u), y(u), z(u)\,)$ lays in the plane $x - 3y + 2z = 0$,

d. the triple $(\,x(u), y(u), z(u)\,)$ tends to the infinity as u approaches the zeroes z_1 and z_2, namely the limit is $\infty\cdot(1, 1, -1)$, $\infty\cdot(-1, -1, 1)$, $\infty\cdot(-1, 1, 1)$ and $\infty\cdot(1, -1, -1)$ as u tends to z_1 and to z_2 from the left side and from the right side, resp.

Points $(1, 1, -1)$, $(-1, -1, 1)$, $(-1, 1, 1)$ and $(1, -1, -1)$ may be interpreted as the vectors parallel to corresponding straight lines, and our $(1, b)$-curve is still closer and closer to these lines as the parameter u approaches the zeros z_1 and z_2. One can get an idea of this situation when looks at Fig.6.1.b. It shows the projections of our $(1, b)$-curve traced on the basic planes of the system *Oxyz*, i.e., on the planes *Oxy*, *Oxz*, *Oyz*.

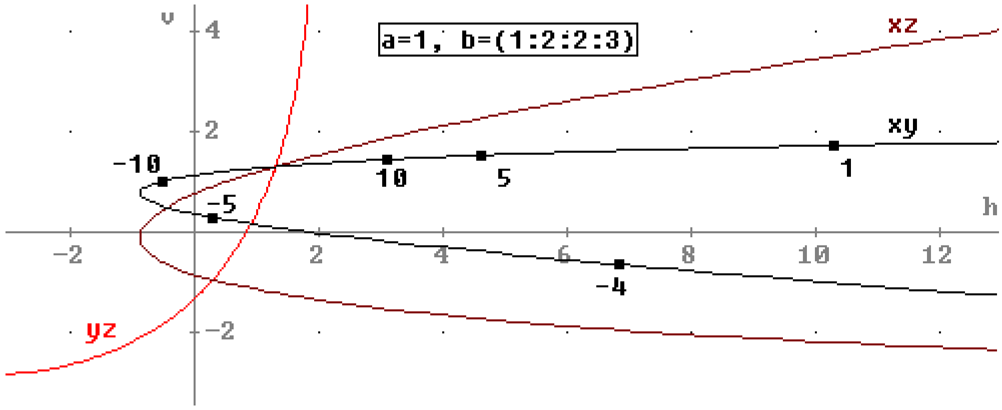

Figure 6.2: Hyperbolas *xy*, *xz*, *yz* governed by the equations $(h, v) = (x(u), y(u))$, $(h, v) = (x(u), z(u))$ and $(h, v) = (y(u), z(u))$ when $a = 1$, $b = (\,1 : 2 : 2 : 3\,)$. On the curve *xy* there are marked points obtained for $u = -4, -5, -10$ and $u = 1, 5, 10$, all they are taken from intervals $(-\infty, z_1)$ and $(z_2, +\infty)$. Points produced for $u \in (z_1, z_2) \approx (-2.63507, -2.31492)$ form the second arm of hyperbola *xy*, in Figure it is not out of the scope, its vertex is at the point appr. $(-1635, -62.5)$

6.2 Second hyperbolical example

For $b = (\,1 : 10 : 2 : 3\,)$ we have a hiperbola described by equations

$$x(u) = \frac{26u^2 + 511u + 1943}{20u^2 + 99u + 122},\; y(u) = \frac{26u^2 + 167u + 222}{20u^2 + 99u + 122},\; z(u) = \frac{26u^2 + 165u + 554}{20u^2 + 99u + 122},$$

where $u \in (-\infty, z_1) \cup (z_1, z_2) \cup (z_2, +\infty)$,

z_1 and z_2 are zeros of the equation $\psi_4(u) = 0$, i.e.,

$$z_1 = \frac{-99 - 2\sqrt{41}}{40} \approx -2.63507,\; z_1 = \frac{-99 + 2\sqrt{41}}{40} \approx -2.31492.$$

Now the combination (19) is the linear system

$$\begin{bmatrix} 26 & 26 & 26 \\ 511 & 167 & 222 \\ 26 & 265 & 554 \end{bmatrix} \cdot \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 20 \\ 99 \\ 122 \end{bmatrix} \cdot D.$$

Its solution is $A = \frac{32512}{353925}\cdot D$, $B = \frac{306841}{235950}\cdot D$, $C = \frac{-441047}{707850}\cdot D$, so the resulting curve lays in the plane $65024\cdot x + 920523\,y - 441047\,z = 707850$ (see Fig.6.2).

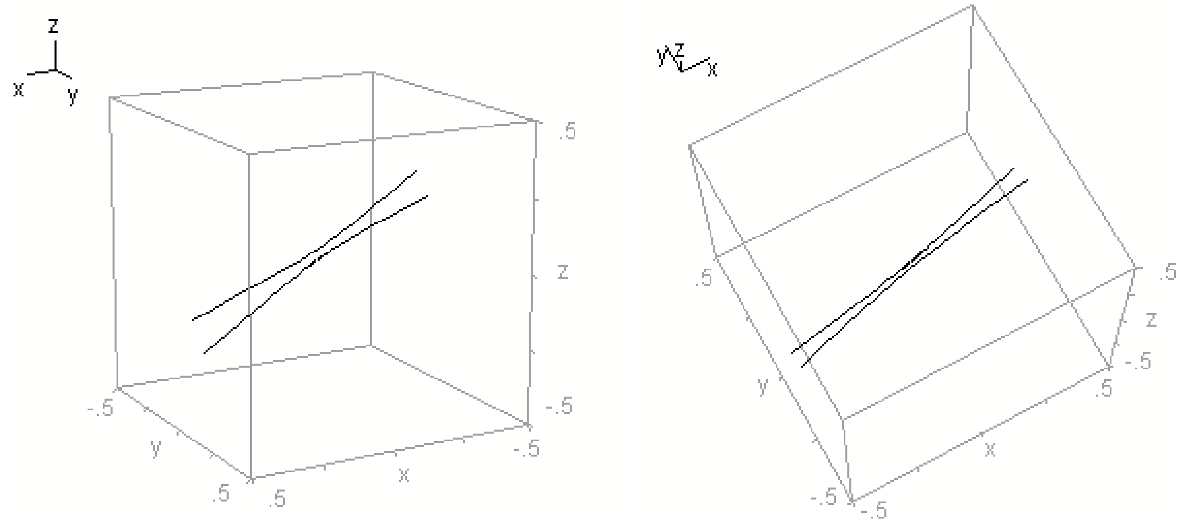

Figure 6.3a, b: The part of (1, b)-curve laying in the cube $<-0.5, 0.5>^2$, where $a = 1$, $b = (3{:}2{:}2{:}3)$, seen from two different points of observation

6.3 Third hyperbolic example

Setting $b = (\,3 : 2 : 2 : 3\,)$ we have (1, b)-curve described by equations
$\psi_1(u) = -2\cdot(3u+7)$, $\psi_2(u) = -2\cdot(u+2)$, $\psi_3(u) = 2\cdot(u+2) = -r_1(u)$, $\psi_4(u) = 6\cdot(2u^2 + 11u + 10)$.
In $\boldsymbol{R}^3$ it covers the hyperbola (see Fig.6.3) parametrized by $u \in (-\infty, z_1) \cup (z_1, z_2) \cup (z_2, +\infty)$, where $z_1 \approx -4.35078$ and $z_2 \approx -1.14921$ are zeroes of ψ_4.
The combination (19) gives the system which is incompatible if $D \neq 0$. For $D = 0$ its solution is $(A, B, C) = (0, B, B)$, where B is any non-zero real, so we get the equation $y + z = 0$. It says the (1, b)-curve lays in the plane $y + z = 0$.

Let's deal, for a moment, with arbitrary $a \neq 0$ and, as above, with $\lambda = -1$ and $b = (\,3 : 2 : 2 : 3\,)$. Now (19) yields the system

$$\begin{bmatrix} -14 & -6\cdot(3a^3-6a^2+2a+2) & \alpha \\ -4 & -2\cdot(3a^3-20a^2+17a-2) & a\cdot\alpha \\ 4 & -2\cdot(2a^3-17a^2+20a-62) & a^2\cdot\alpha \end{bmatrix}\cdot\begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 6a^3\cdot(a^3-2a^2+4a-1) \\ 6\cdot(2a^3+12a^2-6a+3) \\ 60 \end{bmatrix}\cdot D,$$

where $\alpha := 6\cdot(\,a^3 - 2a^2 + 2a - 1\,)$. The only real value a making the quantity α vanishes is 1. This is an other verification that it is enough to work for $a = 1$.

6.4 Extremely simple hyperbolic example

We act again with $\lambda = -1$ and $a = 1$. In the relation (15) for $b = (\,-1{:}\ 1{:}\ 1{:}\ 1)$ there is

$$\psi_1(u) = -4\cdot(u^2 + 4u + 3),\ \psi_2(u) = -4\cdot(u^2 + 2u + 2) = \psi_3(u) = \psi_4(u),$$

and the natural representation in $\boldsymbol{R}^3$ is (1, b)-curve governed by the equation

$$x(u) = (u+3)/(u+1),\ y(u) = 1,\ z(u) = 1,\ u \in (-\infty, -1) \cup (-1, +\infty).$$

6.5 Elliptical example

For $b = (1{:}\ 2{:}\ -3{:}\ 1)$ we have $\psi_1(u) = -2\cdot(15u^2 + 48u + 35)$, $\psi_2(u) = -2\cdot(15u^2 - 9u - 28)$,

$$\psi_3(u) = -2\cdot(15u^2 + 24u - 7),\ \psi_4(u) = 2\cdot(9u^2 + 30u + 29) > 0,$$

so the (1, b)-curve is an ellipse. The equation of this projection upon $z = 0$ is (see Fig.6.5)

$$9140x^2 + 1508x\cdot y + 458y^2 + 14277x + 4674y + 735 = 0.$$

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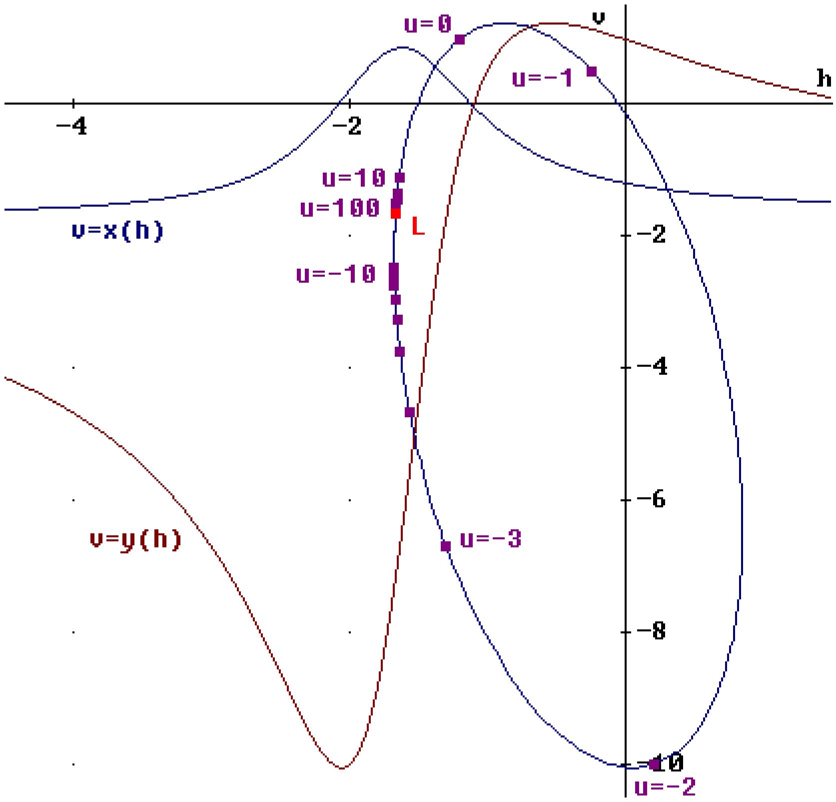

Figure 6.5: Graphs $v = x(h)$, $v = y(h)$, where $x(h) := -\frac{15h^2+48h+35}{9h^2+30h+29}$, $y(h) := -\frac{15u^2-9u-28}{9u^2+30u+29}$, and the ellipse covered by the relation ($h = x(u)$, $v = y(u)$), $-\infty < u < +\infty$. There are also marked points btained for u = –10, –3, –2, 0, 10, 100 and the limiting point L = (–5/3, –5/3)

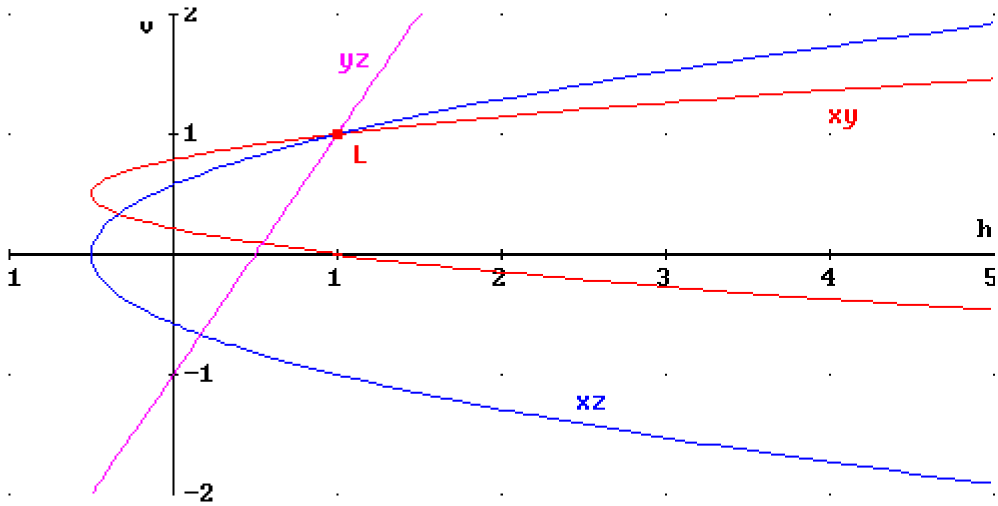

Figure 6.6: Projections of the parabola discussed in Example 6.6; as always, denotations *xy*, *xz* and *yz* stand for the projections of this curve upon the plane $z = 0$, $y = 0$ and $x = 0$, respectively

6.6 Parabolic example

For $b = (1:2:1:1)$ we obtain $\psi_1(u) = -2\cdot(3u^2 + 12u + 11)$, $\psi_2(u) = -2\cdot(3u^2 + 7u + 4)$,
$\psi_3(u) = -2\cdot(3u^2 + 8u + 5)$, $\psi_4(u) = -6\cdot(u+1)^2$,
so the $(1, b)$-curve is a parabola. Solving (19) we find that it lays in the plane $2y - z = 1$. Notice that it may be directly deduced from the parametric equations

$$y(u) = \frac{\psi_2(u)}{\psi_4(u)} = \frac{3u+4}{3\cdot(u+1)},\quad z(u) = \frac{\psi_3(u)}{\psi_4(u)} = \frac{3u+5}{3\cdot(u+1)},\ u \neq 1.$$

7 Conclusions and final remarks

In the paper there are discussed curves defined by the condition $(A, B; M, X) = \lambda$, where A lays on a twisted cubic curve K in the projective space $\boldsymbol{P}^3$, B does not, M runs the line AB, X completes the triple (A, B, M) of points to have the cross-ratio equal to a given number λ. There is obtained the explicit formula for these curves. It follows that they are conics for $\lambda = -1$ and a couple of illustrative examples, produced with a CAS (Derive 5 from Texas Instruments, Inc., USA), are presented. There are still not recognized types of resulting curves for $\lambda \neq -1$.

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WIZUALIZACJA OBRAZÓW PROSTYCH W PEWNYM PRZEKSZTAŁCENIU REALIZOWANYM ZA POMOCĄ DWUSTOSUNKU KORZYSTAJĄCA Z SYSTEMU ALGEBRY KOMPUTEROWEJ

Niech K będzie krzywą przestrzenną rzędu trzeciego w przestrzeni rzutowej P 3 i niech M będzie dowolnym punktem tej przestrzeni nieleżącym na K. W wiązce prostych, której wierzchołkiem jest M, znajduje się dokładnie jedna bisekanta. Punkty, w których przecina ona krzywą K, oznaczamy przez T1 i T2. Tematem pracy jest zbadanie miejsc geometrycznych punktów X Î T1T2, dla których dwustosunek (T1, T2; M, X) = –1, gdy punkt M przebiega prostą, którą wyznaczają ustalone punkt krzywej K i punkt, który na K nie leży. Badanie to przeprowadzamy przy użyciu programu Derive 5 for Windows (Texas Instruments, Inc.).