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LIMAÇON OF PASCAL AS AN ANAMORPHIC IMAGE OF A CIRCLE

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Abstract. The content of this paper relates to cylindrical reflective anamorphs. In particular, a horizontally positioned circle on a cylindrical surface will be transformed into a Limaçon of Pascal. Graphical and analytical proof of the construction has been provided.

Keywords: Limaçon of Pascal, reflective anamorphic images (=mirror anamorphs), cylindrical anamorphs

1 Introduction

To the "mirror" or "reflective anamorphs" listed in the previous work [3,6,7,8,9,10] belong to the group of cylindrical anamorphs. To define this type of transformation we need to set a cylinder of revolution with a reflective surface and the proper observation point (=view-point [1,2]), which will be labeled with O^b (Fig. 1). In order to discuss the properties of the discussed transformation two orthographic views: a front view and a top view of this situation will be provided in Fig.1 and Fig.2. The authors will prove that a Limaçon of Pascal is an image of a horizontally positioned circle in a reflective cylindrical transformation.

2 Mapping of a circle belonging to the cylinder's surface for a reflective convex cylindrical anamorph

In order to better understand an anamorphic image of any ellipse in the reflective cylindrical transformation let us first explain transformation of a horizontally positioned circle, which belongs to the cylinder's surface.

Described above reflective cylinder of revolution has been cut with a horizontal plane γ in a circle (Fig.1). Let us construct a reflective anamorph of the circle.

In order to ensure the continuity of transformation let us assume that the cylinder's surface is semi-transparent. It means that the cylinder's surface allows reflecting objects in the internal part of the cylinder surface. These reflections [4,5] are restricted only to singular reflections (no multiple reflections allowed). An anamorphic image of the circle will be determined point-by point as mapping of a series of points, which all belong to the cylinder's surface (Fig.1). For the previously made assumptions it is easy to prove that the circle's image is a curve of the fourth order. Analysis of the construction presented in Fig.1 prompts us to conclude that we have obtained a Limaçon of Pascal. Let us now discuss its construction.

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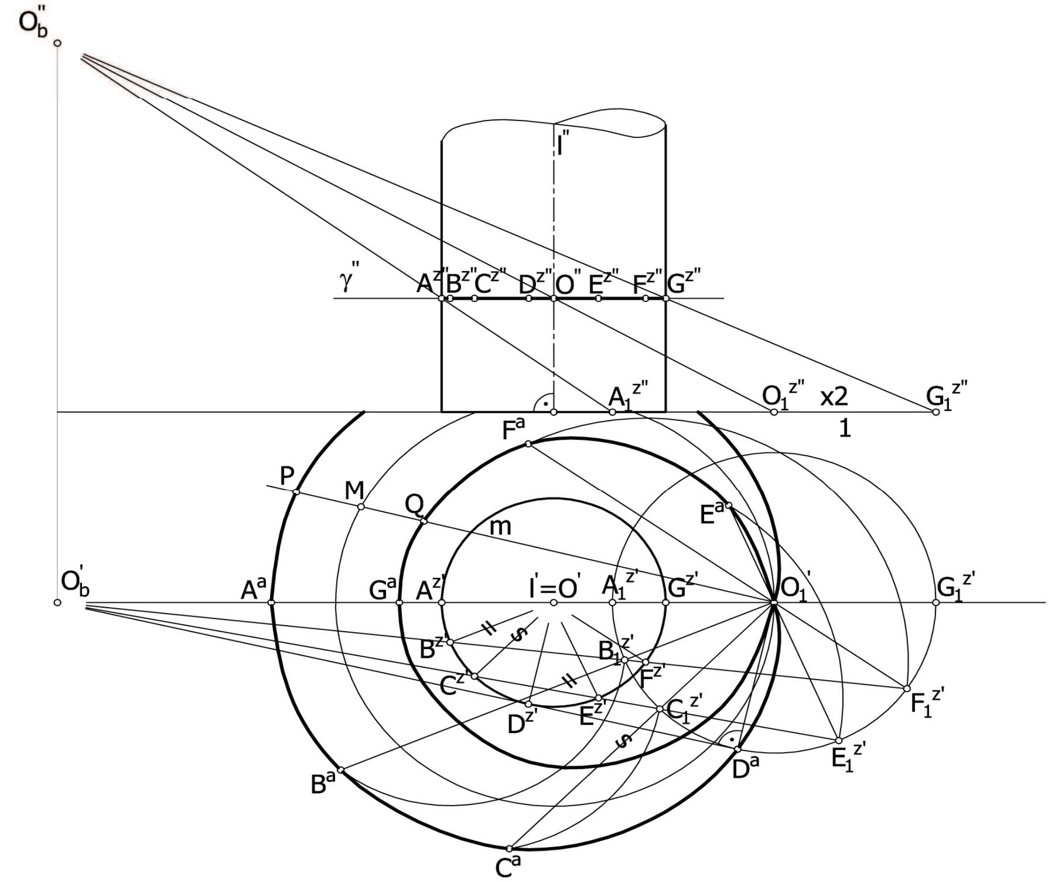

Figure 1: Graphical construction and mapping of a circle belonging to the cylinder's surface in a reflective cylindrical anamorph

Let the circle *k* with center *O'* and radius (*O'O*$_1$*'*) be a directing circle of the obtained curve. Let us measure two equal segments *PM*=*MQ*=*const* on the secant *m* passing through point O_1*'*, while point *M* lies on the circle *k*. The segment *PQ* is equal length with the segment A^aG^a ($PQ=A^aG^a$). Thus the anamorphic image of a circle can be defined as a set of points *PQ* while the secant *m* revolves around point O_1*'*. We have constructed a limaçon of Pascal.

3 Analytical analysis of the curve, which describes an anamorphic image of a circle belonging to the cylinder's surface

Let us assume that the center of projection O_b (= a view point, an observation point) is a real point in a 3D space, while the radius of the cylinder's base is constant and equal to *r* (Fig. 2). On the cylinder's surface let us distinguish a circle, which lies in the horizontal plane γ (Fig.1). At first we will consider transformation of an optional point B^Z of the circle that lies on the reflectively active part of the cylinder. Projection lines connecting the view point O_b with point B^Z and with the center of the circle *O* have been shown in two orthographic views (Fig. 2).

Let us label with $O^Z{}_1$ and $B^Z{}_1$ respectively the points of intersection of the rays *m* and *t* with the projection plane and with B^a the reflective image of point B^Z. The normal line *n* to the cylinder surface at point B^Z is a horizontally positioned line which intersects with the axis of the cylinder *l* at point *R*. According to the law of reflection, the angle of incidence φ defined with the incident ray *t* and the surface normal *n* equals the angle of reflection φ_1 defined with the reflected ray t_1 and the normal *n*. The angles φ and φ_1 are equal and coplanar. Let us notice that the segments B^ZB^a and $B^ZB^Z{}_1$ are equal length. Let us distinguish an optional point N on the normal n. The equality of the angles O_bB^ZN i NB^ZB^a entails the following equality

$$\angle \mathrm{NB^Z\, B^a} = \angle \mathrm{OB^ZB^Z{}_1} = \angle \mathrm{B^ZB^Z{}_1B^a}$$

Hence, we conclude on the following relations

$$\mathrm{O^Z{}_1\, B^Z{}_1} \parallel \mathrm{OB^Z}$$
$$\mathrm{B^Z{}_1B^a} \parallel \mathrm{OB^Z}$$

Thus the points $O^Z{}_1$, $B^Z{}_1$ *and* B^a are collinear.

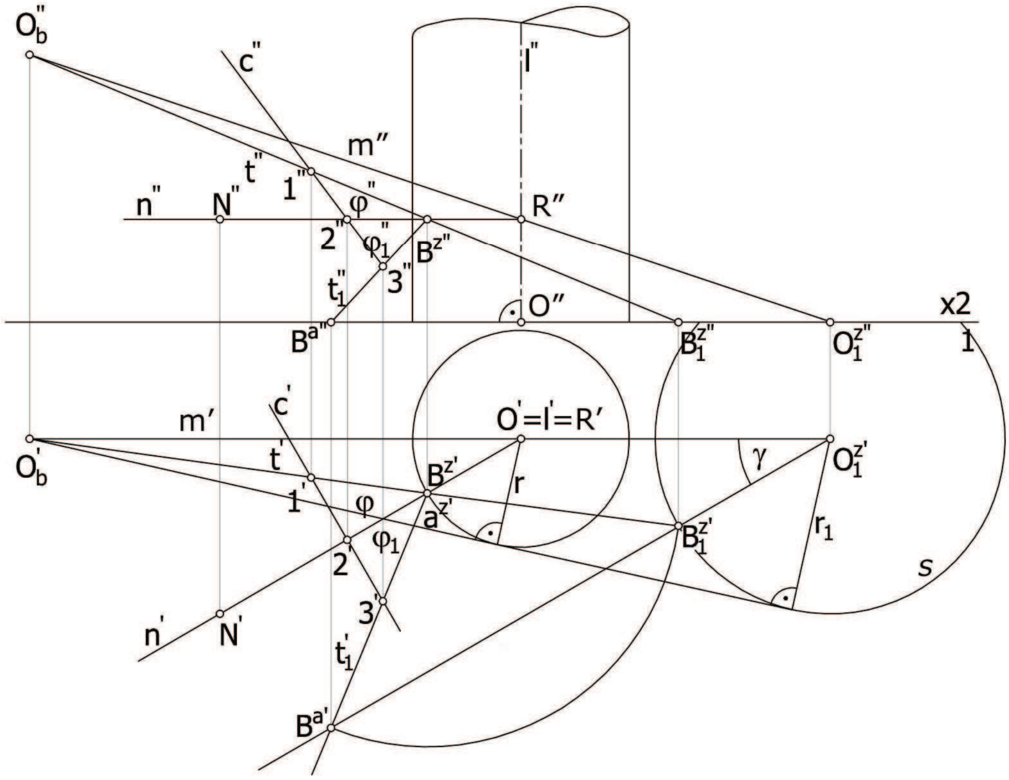

Figure 2: Mapping of a circle belonging to the cylinder's surface in a reflective cylindrical anamorph: Labels used for analytical proof

At the center $O_1{}^{z\,\prime}$ of the circle s let us define a new origin of a polar coordinate system. Let us designate with a symbol "a" the segment $OO^Z{}_1 = a$. The negative semi-axis Ox of the Cartesian coordinate system will become a positive axis of a polar coordinate system. Point $O_1{}^z$ will become the pole of this local system. In the new polar coordinates, the anamorphic images of points will make a geometrical set of points whose coordinates will be expressed with the equation:

$$\mathrm{R}(\gamma) = O^Z{}_1\, B^Z{}_1 + B^Z{}_1B^a = \pm\,(a\cos\gamma + r + r + a\cos\gamma - r_1), \qquad (1)$$

where $\gamma \in <\tfrac{1}{2}\pi\,,\, \tfrac{3}{4}\pi>$ and r_1 is the radius of the circle s.

By simplification of the equation (1) we obtain:

$$\mathrm{R}(\gamma) = \pm\,(2a\cos\gamma + 2r - r_1) \text{ for } \gamma \in <\tfrac{1}{2}\pi\,,\, \tfrac{3}{4}\pi> \qquad (2)$$

Equation (2) is in a specific equation of a conchoid of the circle $k((-a,0),a)$, which is also called a limaçon of Pascal [11]. From the above we conclude that in the described mapping, the image of the circle s is a conchoid of a circle k.

The reasoning and the proof will be analogical for the case of transformation of a circle with the assumption that the view point $O_b{}^{\infty}$ is placed at infinity.

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ŚLIMAK PASCALA JAKO OBRAZ W ANAMORFICZNYM PRZEKSZTAŁCENIU OKRĘGU

Niniejsza praca dotyczy przypadku zwierciadlanych anamorfoz refleksyjnych – powierzchniowych. W szczególności przeprowadzono graficzny i analityczny dowód pewnej własności przekształcenia anamorficznego, z którego wynika, iż obrazem poziomego okręgu leżącego na powierzchni zwierciadła walcowego jest krzywa 4-go rzędu, a konkretnie jest nią ślimak Pascala.