GEOMETRIC MODELLING OF KNOTTED TORI

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Abstract: The knotted torus will be understood as a regular surface created by a knotting of a torus like surface in the three-dimensional space. Knotted torus is a surface with no self-intersections and singular points, it is closed, and its envelope is again a torus. This curve is in the general case trajectory of the specific composite revolutionary movement about two skew axes (interior axis ^1o and exterior axis ^2o) called Euler movement known as the Euler trajectory. Torus is created with nodes analogous to the creation of surface (not solids), by "wrapping" of the torus in E3. Torus knots with a surface that contains no points particular. The author defines the areas considered, present in the work of different types of trajectories Euler, and then classifies torus three types of nodes depending on the location of the base circle and the position of two axis of rotation. The work was illustrated sample images of the three types of torus knots.

Keywords: torus, knotted tori, Euler movement, Euler trajectory.

In the presented paper, the knotted torus will be understood as a regular surface created by a knotting of a torus like surface (not solid) in the three-dimensional space $\mathbf{E}^3$. Knotted torus is a surface with no self-intersections and singular points, it is closed, and its envelope is again a torus. This means that by knotting a torus no new type of topological structure will be created, or no new type of knot will arise than the original one. Consequently, there exist infinitely many different forms of the knotted tori in the three dimension space.

From the geometric point of view we can assume that a knotted torus is an envelope surface created by the continuous movement of the sphere in the space, whereas the trajectory is a curve located on a torus of revolution in such way that no self-intersections might occur. This curve is in the general case trajectory of the specific composite revolutionary movement about two skew axes (interior axis ^1o and exterior axis ^2o) called Euler movement known as the Euler trajectory and described in details in [1]. Simple Euler trajectory in the basic form is a closed space curve without any multiple points; its orthographic view in the plane perpendicular to the axis of revolution ^2o is a symmetric planar curve Limaçon of Pascal. Euler trajectory is located on torus of revolution with the axis in the exterior axis of revolution ^2o, and radius equal to the distance a of the moving point from the interior axis of revolution ^1o. Suppose, axis of the inner revolution is located in the coordinate axis z and axis of the outer revolution is line parallel to the coordinate axis x in the distance d, then coordinates of the Euler trajectory points satisfy the implicit equation of the torus in the form

$$\left(x^2 + (y-d)^2 + z^2 - a^2\right)^2 = 4d^2\left((y-d)^2 + z^2\right).$$

Several examples of different forms of Euler trajectory determined by multiples k and l of angular velocities of the two axial revolutions are illustrated in fig. 1 on the left, in the middle and on the right, with pairs of parameters $a = 5$, $d=3$, $(k, l) = (2,2), (2,6), (4,6)$, respectively. Orthographic views of the curves to the plane perpendicular to the outer axis of revolution ^2o are symmetric cycloidal curves presented in the second row.

There can be distinguished 3 different types of knotted tori, with respect to the position of the basic circle and the two axes of revolutions. Shaping parameters – angular velocities of the two simultaneous revolutions – define the specific forms of these surfaces.

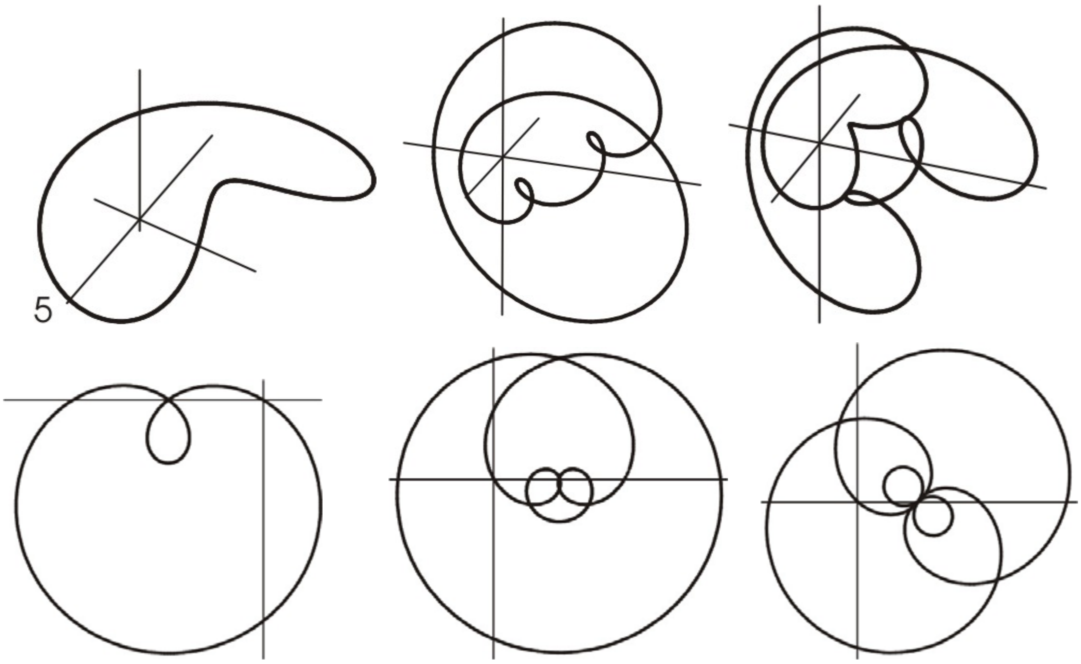

Fig. 1: Euler trajectories

Knotted tori of the first type are surfaces in the group of cyclical toroidal two-axial surfaces of revolution of Euler type (complete classification is presented in [2]), which can be generated by the movement of the basic circle g located in the plane passing through the interior axis of revolution ${}^{1}o$ that is determined by the vector equation for $u \in \langle 0,1 \rangle \subset R$

$$r(u) = (a + r\cos 2\pi u, 0, r\sin 2\pi u, 1).$$

Parametric equations of surfaces in this subgroup are for $(u,v) \in \langle 0,1 \rangle^2$ in the form

$$\begin{aligned} x(u,v) &= (a + r\cos 2\pi u)\cos k\pi v \\ y(u,v) &= (a + r\cos 2\pi u)\sin k\pi v \cos l\pi v - r\sin 2\pi u \sin l\pi v + d(\cos l\pi v - 1) \\ z(u,v) &= (a + r\cos 2\pi u)\sin k\pi v \sin l\pi v + r\sin 2\pi u \cos l\pi v + d\sin l\pi v \end{aligned}$$

These surfaces consist of several threads and knots and they can be created by choosing values of shaping parameters satisfying the following relations: $r < a < d$, k and l are even numbers, whereas none is the multiple of the other. Different modifications can be achieved by the choice of the multiples of angular velocities k and l. Number of windings about the second axis of revolution is the number of arms equal to $l/2$, number of windings about the first axis of revolution defines the number of knots, $k/2$. Simple non-trivial trefoil is illustrated in fig. 2 on the left, its shaping characteristics are $a = 5$, $d = 10$, $r = 3$, $(k, l) = (6, 4)$, for 2 arms knotted 3 times. Other two forms of knotted tori are determined by the following values: $a = 5$, $d = 12$, $r = 3$, $(k, l) = (14, 6)$ for 3 arms knotted 7 times, and $a = 6$, $d = 12$, $r = 3$, $(10, 8)$ for 4 arms knotted 5 times.

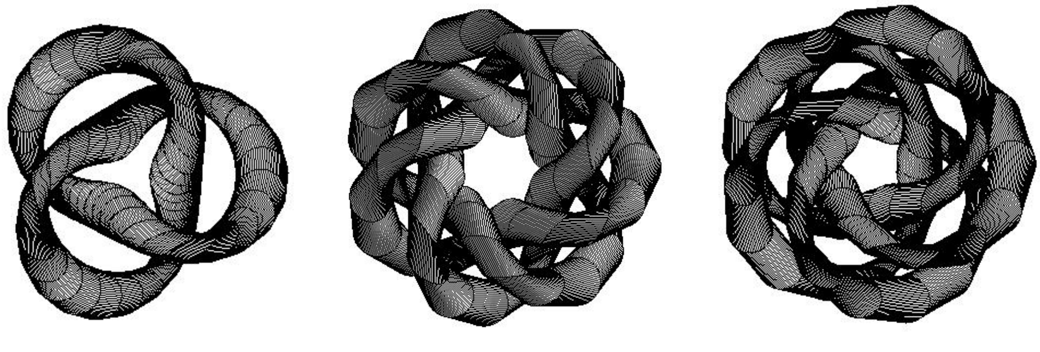

Fig. 2: Knotted tori of the first type

Knotted tori of the second type are modelled as cyclical normal two-axial surfaces of revolution of Euler type. Basic circle of these knotted tori is located in the plane passing through the exterior axis of revolution ${}^2o$ determined by the vector equation for $u \in \langle 0,1 \rangle \subset R$

$$r(u) = (a + r\cos 2\pi u,\ r\sin 2\pi u, 0, 1).$$

Surfaces consist of multiple knots with more threads cycled arround the outer axis. Surfaces are represented by parametric equations for $(u,v) \in \langle 0,1 \rangle^2$ in the form

$$x(u,v) = (a + r\cos 2\pi u)\cos k\pi v - r\sin 2\pi u \sin k\pi v$$
$$y(u,v) = (a + r\cos 2\pi u)\sin k\pi v \cos l\pi v + r\sin 2\pi u \cos k\pi v \cos l\pi v + d(\cos l\pi v - 1)$$
$$z(u,v) = (a + r\cos 2\pi u)\sin k\pi v \sin l\pi v + r\sin 2\pi u \cos k\pi v \sin l\pi v + d\sin l\pi v$$

Different modifications of knotted tori of the 2nd type can be created by shaping characteristics in multiples of angles of revolutions k and l. Number of turns about the first axis of revolution defines the multiplicity of the knot, which is the number of threads $k/2$. Number of turns about the second axis of revolution gives the number of cycles $l/2$ about the exterior axis of revolution ${}^2o$. Examples of surfaces are illustrated in fig. 3, parameter values for presented forms from left to right are: $a = 10$, $d = 17$, $r = 3$, $(k, l) = (4, 6), (6, 8), (8, 10)$.

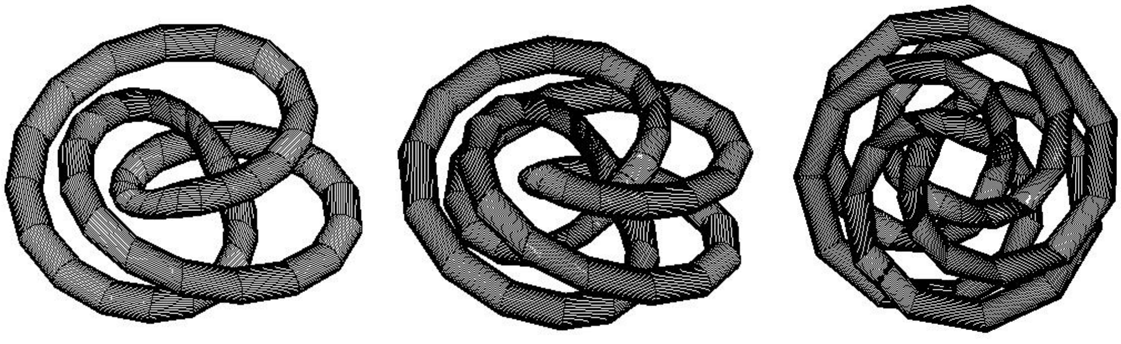

Fig. 3: Knotted tori of the second type

Knotted tori of the third type are surfaces in the group of cyclical general two-axial surfaces of revolution of Euler type; these can be generated by the movement of the basic circle g located in the general position to the two axes of revolutions, i.e. in the plane not passing through any of the two axis of revolutions, interior axis ${}^1o$ or exterior axis ${}^2o$. Considering the basic circle determined by the vector equation for $u \in \langle 0,1 \rangle \subset R$

$$r(u) = (a + r\cos 2\pi u,\ b + r\sin 2\pi u, c, 1)$$

parametric equations of these surfaces are for $(u,v) \in \langle 0,1 \rangle^2$ in the form

$$x(u,v) = (a + r\cos 2\pi u)\cos k\pi v - (b + r\sin 2\pi u)\sin k\pi v$$
$$y(u,v) = (a + r\cos 2\pi u)\sin k\pi v \cos l\pi v + (b + r\sin 2\pi u)\cos k\pi v \cos l\pi v - \\ - c\sin l\pi v + d(\cos l\pi v - 1)$$
$$z(u,v) = (a + r\cos 2\pi u)\sin k\pi v \sin l\pi v + (b + r\sin 2\pi u)\cos k\pi v \sin l\pi v + \\ + c\cos l\pi v + d\sin l\pi v$$

Different modifications of knotted tori of the 3rd type differ from the surfaces of the 2nd type in the form, which is determined by the values of parametes *b* and *c*. Number of turns about the first axis of revolution defines the number of arms *k*/2 (multiplicity of the knot), number of turns about the second axis of revolution gives the number of cycles *l*/2. Examples of surfaces in fig. 3 are defined by values of parameters: *a* = 10, *b* = 5, c = 5, *d* = 17, *r* = 3, (*k*, *l*) = (4, 6) on the left, *a* = 10, *b* = 5, c = 8, *d* = 17, *r* = 3, (*k*, *l*) = (4, 10) in the middle, and *a* = 10, *b* = 7, c = 5, *d* = 17, *r* = 3, (*k*, *l*) = (6, 8) on the right.

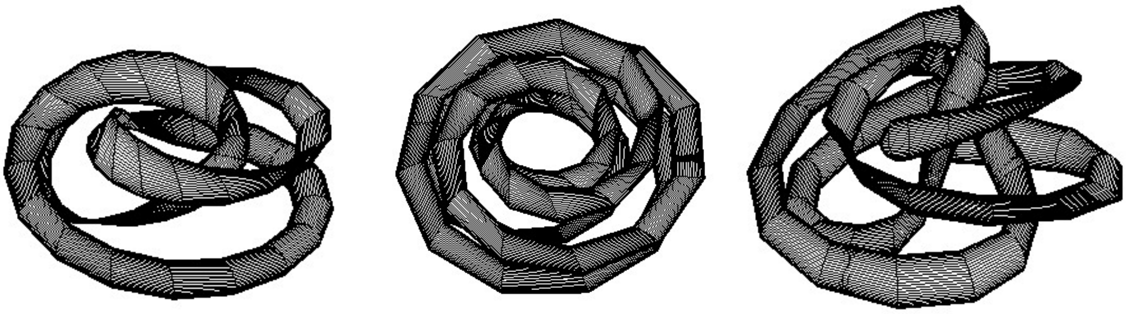

Fig. 4: Knotted tori of the third type

References

[1] D. Velichová: *Trajectories of composite rotational movements*, G – Slovak Journal for Geometry and Graphics, ISSN 1335- X, Vol. 3, No. 5, 2006, pp. 47-64.

[2] D. Velichová: *Classification of two-axial surfaces*, G – Slovak Journal of Geometry and Graphics, ISSN 1335- X, Vol. 4, No. 7, 2007, pp. 63-82.

GEOMETRYCZNE MODELOWANIE TORUSÓW Z WĘZŁAMI

Pod pojęciem „torus z węzłem” należy rozumieć taką zamkniętą, regularną powierzchnię, która jest obwiednią powierzchni tworzonej przez ruch ciągły sfery w przestrzeni. Trajektoria tego ruchu jest krzywą, która nie posiada punktów osobliwych i przynależy do powierzchni torusa obrotowego. W ogólnym przypadku krzywa ta jest trajektorią specyficznego, złożonego ruchu obrotowego wokół dwóch prostych skośnych (osi wewnętrznej ${}^1o$ i osi zewnętrznej ${}^2o$), znanego w literaturze pod nazwą ruchu Eulera. Torus z węzłami jest tworzony analogicznie do tworzenia powierzchni (a nie brył), przez „zawijanie” torusa w przestrzeni $\mathbf{E}^3$. Torus z węzłami jest powierzchnią nie zawierającą punktów osobliwych. Autorka definiuje rozpatrywane powierzchnie, przedstawia w pracy różne typy trajektorii Eulera, a następnie klasyfikuje trzy typy torusów z węzłami w zależności od położenia bazowego okręgu oraz położenia dwóch osi obrotu. Praca została zilustrowana przykładowymi obrazami trzech typów torusa z węzłami.