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CONSTRUCTION OF STRAIGHT LINE INTERSECTING FOUR GIVEN STRAIGHT LINES

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Abstract: The problem of constructing a straight line intersecting four given lines skew to each other is presented in Monge's projections. The solution consists in choosing an adequate system of projection planes and applying the Steiner's construction of common lines of two projective pencils of lines.

Keywords: pencil of lines, range of points, projectivity, perspectivity, Steiner's construction.

1. Introduction

Consider four arbitrary straight lines *a, b, c, d* skew to each other.

The problem of constructing the straight line intersecting each of these lines will be solved by the Monge's projections method.

First suppose that none of these lines is ideal. In this case, without diminishing the generality of assumptions, we can take the horizontal projection plane π_1 parallel to two lines, for example *a* and *d,* and the vertical projection plane π_2 perpendicular to one of these lines.

So we can admit that the Monge's projections of lines *a, b, c, d* are given, when *a* is vertically projecting, *d* is horizontal, *c* and *b* are in general position.

The symbols $\barwedge$ and $\doublebarwedge$ will be used to denote the projectivity and the perspectivity, respectively.

2. Analysis and Construction

On each line *b* and *c* we choose three arbitrary points X_b , Y_b , Z_b and X_c , Y_c , Z_c respectively such that the lines determined by the pairs $X_b''X_c''$, $Y_b''Y_c''$, $Z_b''Z_c''$ of their vertical projections pass through the point $P = a''$ (Fig.1). Now we determine the horizontal projections of the chosen points as well as the projections X_b''', Y_b''', Z_b''' and X_c''', Y_c''', Z_c''' on a auxiliary projection plane perpendicular to the line *d*. The lines x_b , y_b , z_b pass through the point $Q = d'''$ and the points X_b''', Y_b''', Z_b''' respectively and the lines x_c , y_c , z_c pass trough the point *Q* and the points X_c''', Y_c''', Z_c''' respectively.

Let us now take into consideration some projective relations on the picture plane.

We have two ranges of points $(X_b''', Y_b''', Z_b''', \ldots)$ and $(X_c''', Y_c''', Z_c''', \ldots)$ with the bases b''' and c''' respectively. It is easy to see that

$$(X_b''', Y_b''', Z_b''', \ldots) \barwedge (X_b'', Y_b'', Z_b'', \ldots) \text{ and } (X_c''', Y_c''', Z_c''', \ldots) \barwedge (X_c'', Y_c'', Z_c'', \ldots)$$

Because the ranges $(X_b'', Y_b'', Z_b'', \ldots)$ and $(X_c'', Y_c'', Z_c'', \ldots)$ are defined in the way as they are in the perspectivity relation we obtain as well

$$(X_b''', Y_b''', Z_b''', \ldots) \barwedge (X_c''', Y_c''', Z_c''', \ldots) \tag{1}$$

For the pencils of lines $(x_b , y_b , z_b , \ldots)$ and $(x_c , y_c , z_c , \ldots)$ with the common vertex *Q* we have the following relations:

$$(x_b , y_b , z_b, \ldots) \doublebarwedge (X_b''', Y_b''', Z_b''', \ldots) \text{ and } (x_c , y_c , z_c, \ldots) \doublebarwedge (X_c''', Y_c''', Z_c''', \ldots) \tag{2}$$

Hence it follows from the properties (1) and (2) that

$$(x_b , y_b , z_b , \ldots) \barwedge (x_c , y_c , z_c , \ldots) \tag{3}$$

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Now we can apply the Steiner's construction to find the common lines of the projective pencils (x_b , y_b , z_b , ...) and (x_c , y_c , z_c , …).
To simplify the construction we take an arbitrary circle S^2 which passes trough the point Q. Points of intersection of this circle with the lines determining the considered pencils are denoted by B_x , B_y , B_z and C_x , C_y , C_z respectively (Fig.1).

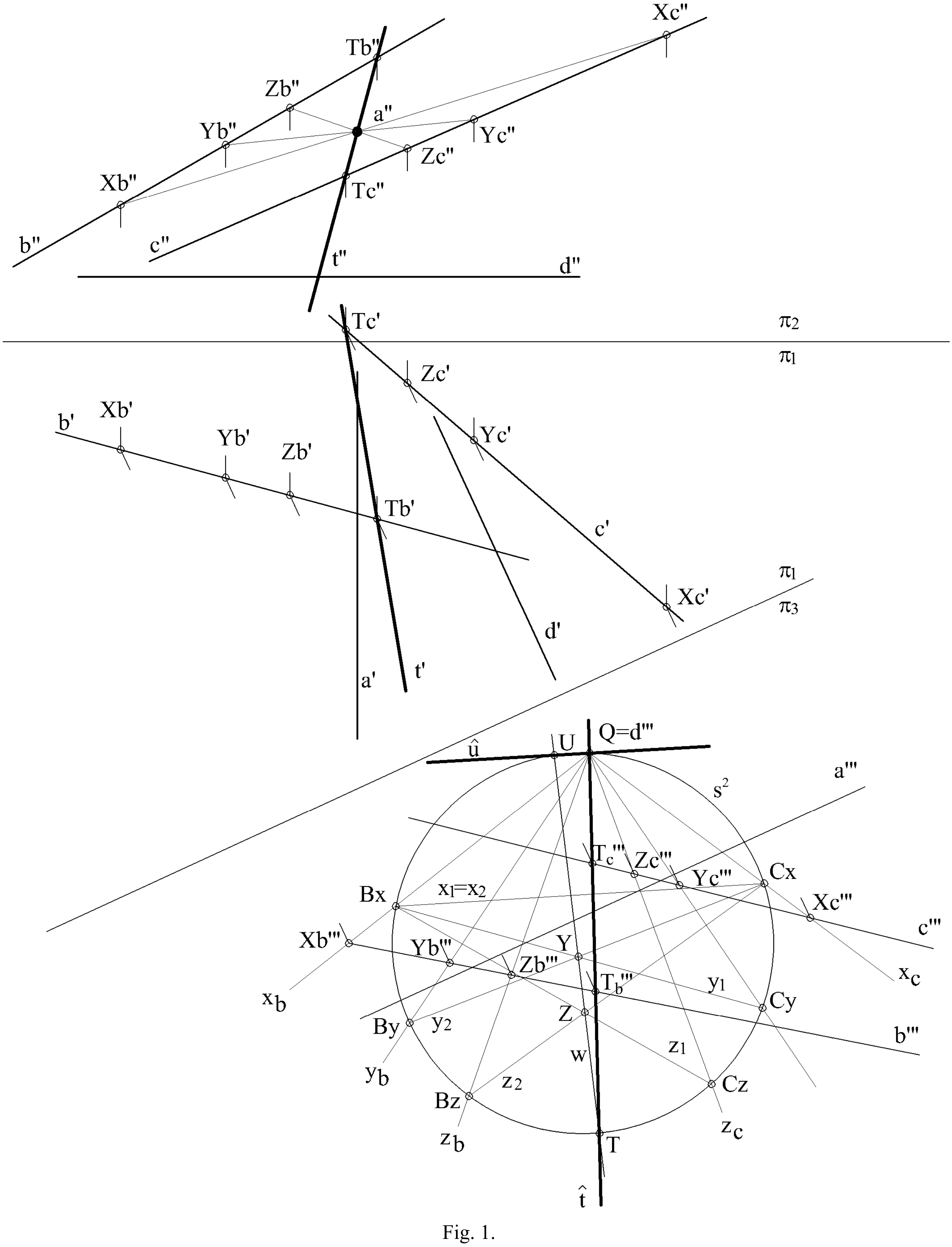

Fig. 1.

According to the Steiner's construction (see for example [2] p.156) we consider additionally two other pencils of lines: the pencil $(x_1, y_1, z_1, \ldots)$ with the vertex B_x when the lines x_1, y_1, z_1 are determined by the points C_x, C_y, C_z respectively and the pencil $(x_2, y_2, z_2, \ldots)$ with the vertex C_x and the lines x_2, y_2, z_2 determined by the points B_x, B_y, B_z respectively.
From the properties of conics as projective loci (see [3], ch.IV §5) we obtain the following relations:

$$(x_b, y_b, z_b, \ldots) \barwedge (x_2, y_2, z_2, \ldots) \quad \text{and} \quad (x_c, y_c, z_c, \ldots) \barwedge (x_1, y_1, z_1, \ldots) \tag{4}$$

An immediate implication from the relations (3) and (4) is that

$$(x_1, y_1, z_1, \ldots) \barwedge (x_2, y_2, z_2, \ldots)$$

The above projective pencils are also perspective because they have the common line $x_1 = x_2$.

The line w defined by the points $Y = y_1 \wedge y_2$ and $Z = z_1 \wedge z_2$ intersects the circle S^2 at the points T and U. Then the lines $\hat{t}\,(Q,T)$ and $\hat{u}\,(Q,U)$ are the common lines of the projective pencils $(x_b, y_b, z_b, \ldots)$ and $(x_c, y_c, z_c, \ldots)$ with the common vertex Q.

On the lines $\hat{t}$ and $\hat{u}$ are lying the pairs of the corresponding points $T_b'''T_c'''$ and $T_b'''T_c'''$ of the projective ranges $(X_b''', Y_b''', Z_b''', \ldots)$ and $(X_c''', Y_c''', Z_c''', \ldots)$.

Now we can determine the points T_b'', T_c'' and T_b', T_c' as well as U_b'', U_c'' and U_b', U_c'. It is not difficult to observe that the lines t and u, defined by the projections $t'(T_b', T_c')$, $t''(T_b'', T_c'')$ and $u'(U_b', U_c')$, $u''(U_b'', U_c'')$ respectively, intersect each line a, b, c, d. In Fig.1 only the projections of the line t are drawn.

We have two solutions if the line w cuts the circle S^2 at two points. If the line w is tangent to the circle S^2 we have only one straight line intersecting four given lines. If the line w has no common points with the circle S^2 there is no straight line intersecting simultaneously all given lines.

3. Conclusions

1) If two from the lines a, b, c, d are perpendicular, for example $a \perp d$, then the problem put forward can be solved without using the auxiliary projection plane by taking the horizontal projection plane perpendicular to d. In similar way as presented above the Steiner's construction can be made in the horizontal projection with $Q = d'$. The lines of the projective pencils $(x_b, y_b, z_b, \ldots)$ and $(x_c, y_c, z_c, \ldots)$ are defined by points $(X_b', Y_b', Z_b', \ldots)$ and $(X_c', Y_c', Z_c', \ldots)$ respectively.

2) Observe that among the four given lines a, b, c, d skew to each other no more than one can be ideal. Suppose that the line a is ideal. In this case also the system of two projection planes is sufficient. Namely the horizontal projection plane can be taken perpendicular to d and the vertical projection plane perpendicular to a plane α incident to the ideal line a. Cutting the lines b'' and c'' by the straight lines parallel to the horizontal projection α'' of the plane α we obtain two perspective ranges of points: $(X_b'', Y_b'', Z_b'', \ldots)$ and $(X_c'', Y_c'', Z_c'', \ldots)$ with the bases b'' and c'' respectively. The construction can be continued as in the previous cases.

3) The presented construction can be applied to piercing a ruled surface by a straight line. Suppose that the straight lines a, b, c are directrices of a ruled surface. To find the points in which line d pierces this surface it is enough to determine the elements of this surface which intersect the given line d, i.e. the straight lines intersecting the four lines a, b, c, d. The common points of these elements and the line d are the required points of intersection.

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