SUBSPACE PROJECTIONS WITH CENTRES DISPERSED ON SECOND-DEGREE FORMS

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Abstract. The concept of subspace projections has already been discussed in the subject literature and the rules of operation and properties of the projections whose centres and projecting forms for particular points of represented space are subspaces, were defined in relation to so-called bundle projections (with fixed centres) and to subspace projections with bundle dispersed centres. In this paper some general properties of subspace projections with centres dispersed on second-degree forms are discussed.

Key Words: subspace projection, bundle projection, subspace projection with centres dispersed on second-degree forms.

1. Introduction

The rules of operation and properties of subspace projections, i.e. the projections whose centres and projecting forms for particular points of represented space are subspaces, were already specified in relation to so-called bundle projections (with fixed centres) in [1], and to subspace projections with bundle dispersed centres [2,3]. The paper discusses general properties of subspace projections with centres dispersed on second-degree forms. This type of projection is referred to as PQ.

In order to define the apparatus of projection PQ for an n-dimensional projective $P_n$ space let us establish in this space three bundles $< P_n, K >$, $< F_1, C_1 >$, and $< F_2, C_2 >$ of the same type (i.e. with the same factors of manifolds $f_M = n - \dim K + 1$), with the following properties:

1° $\dim F_1 = \dim F_2$, and $\dim C_1 = \dim C_2$,
2° $F_1 = F_2$, whereas $C_1 \neq C_2$, if the defining apparatus of the PQ projection belongs to the family of so-called meet projections, marked here as PQM (fig.1b),
3° $F_1 \neq F_2$, whereas $C_1 = C_2$, if the defining apparatus of the PQ projection belongs to the family of so-called junction projections, marked here as PQJ (fig.1a).

Moreover, let us introduce the following projective transformations:
- $C_l$ being a collineation that transforms the bundle $< P_n, K >$ onto the bundle $< F_1, C_1 >$,
- $C_r$ being a correlation that transforms the bundle $< P_n, K >$ onto the bundle $< F_2, C_2 >$.

Figure1: Schematic diagram of the structures and the principles of operation of apparatuses:
- a - PQJ subspace junction projections,
- b - PQM subspace meet projections.
With such defined projective transformations, the bundles \(< F_1, C_1 > \) and \(< F_2, C_2 > \) correspond with each one to the other in the correlation \( CR = Cr (C^1) \).

The established bundles and defined projective transformations allow us to associate with every point \( X_i \in P_n \) the following three subspaces: 
1° \( \mathbf{L}_i = X_i \cap K \),
2° \( \mathbf{D}_{\mathbf{R}_i} = \mathbf{C} \left( \mathbf{L}_i \right) \in \langle F_1, C_1 \rangle \),
3° \( \mathbf{D}_{\mathbf{R}_i} = \mathbf{C} \left( \mathbf{L}_i \right) \in \langle F_2, C_2 \rangle \).

Let us also assume that:
- in case of a so-called meet subspace projection \( \mathbf{PQP} \), the meet \( \mathbf{D}_{\mathbf{R}_i} \cap \mathbf{D}_{\mathbf{R}_i} = \mathbf{S}_i \) with the dimension equal to \( \dim \mathbf{C}_1 = \dim \mathbf{C}_2 \),
- in case of a so-called junction subspace projection \( \mathbf{PQJ} \), the junction \( \mathbf{D}_{\mathbf{R}_i} \mathbf{o} \mathbf{D}_{\mathbf{R}_i} = \mathbf{S}_i \) with the dimension equal to \( \dim \mathbf{F}_1 = \dim \mathbf{F}_2 \),

is the centre of projection for a distinguished point \( X_i \in P_n \).

It is easy to prove, that:

- in the \( \mathbf{PQM} \) projection, centres \( \mathbf{S}_i \) of this projection, as meets of homologous elements of the correlated bundles \( \langle F_1, C_1 \rangle \) and \( \langle F_2, C_2 \rangle \) are subspace formers of a second-degree form \( \Sigma_P \),
- in the \( \mathbf{PQJ} \) projection, centres \( \mathbf{S}_i \) of this projection, as junctions of homologous elements of the correlated bundles \( \langle F_1, C_1 \rangle \) and \( \langle F_2, C_2 \rangle \) are subspace generators of a second-degree form \( \Sigma_i \).

Therefore, the defined sets of centres of the projections \( \mathbf{PQM} \) and \( \mathbf{PQJ} \) are subspaces dispersed on the second degree forms \( \Sigma_P \) and \( \Sigma_i \), respectively, and they satisfy our preliminary assumptions made in this analysis.

In order to complete the apparatuses of the projections \( \mathbf{PQM} \) and \( \mathbf{PQJ} \), one should still establish their forms of projections in the \( P_n \) space. Similarly, as in the case of any classical subspace projection also in the case of currently considered projections one should consider as the form of projection such a subspace \( \mathbf{P} \) which determines the represented \( P_n \) space together with almost every one of the centres \( \mathbf{S}_i \). Moreover, depending on whether the defined projection is supposed to be an ordinary or generalized projection [4], the distinguished subspace \( \mathbf{P} \) should be either disjoint with almost all the projection centres, or it should cross every of these centres in a subspace nonnegative dimension not greater than \( \dim \mathbf{P} - 2 \).

The \( \mathbf{PQ} \) projections can have practical application when they lead to a graphical representation of the represented \( P_n \) space. This result is achieved when planes, traditionally referred to as \( \pi \), are adopted as forms of projections of the \( \mathbf{PQ} \) projections. In such a case, from our previous identifications it results that if the \( \mathbf{PQ} \) projection is to be:

- an ordinary projection, then \( \dim \mathbf{C}_1 = \dim \mathbf{C}_2 = n - 3 = \dim \mathbf{S}_i \), when the projection under consideration has characteristics of the \( \mathbf{PQM} \) projection; and dim \( \mathbf{F}_1 = \dim \mathbf{F}_2 = n - 3 = \dim \mathbf{S}_i \), when the \( \mathbf{PQ} \) projection belongs to the \( \mathbf{PQJ} \) family of projections;

- a generalized projection, then \( \dim \mathbf{C}_1 = \dim \mathbf{C}_2 = n - 2 = \dim \mathbf{S}_i \), when the projection under consideration has characteristics of the \( \mathbf{PQM} \) projection; and dim \( \mathbf{F}_1 = \dim \mathbf{F}_2 = n - 2 = \dim \mathbf{S}_i \), when the \( \mathbf{PQ} \) projection belongs to the \( \mathbf{PQJ} \) family of projections,

The structures of \( \mathbf{PQ} \) projection apparatuses that lead to mappings of the \( P_3 \) and \( P_4 \) spaces are presented in drawings in tables 1 and 2 respectively.
Table 1. Structure of apparatuses of the PQ mapping projections of $P_3$ space

<table>
<thead>
<tr>
<th>Type of the PQ projection</th>
<th>$\dim C_i$</th>
<th>$\dim F_i$</th>
<th>Visual drawing of a projection apparatus</th>
</tr>
</thead>
<tbody>
<tr>
<td>PQJ generalized</td>
<td>-1</td>
<td>1</td>
<td><img src="image1" alt="PQJ_generalized_diagram" /></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\Sigma$ - envelope conic</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\Sigma$ - warped quadric</td>
</tr>
<tr>
<td>PQM ordinary</td>
<td>0</td>
<td>2</td>
<td><img src="image2" alt="PQM_ordinary_diagram" /></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\Sigma$ - conic</td>
</tr>
<tr>
<td>PQM ordinary</td>
<td>0</td>
<td>3</td>
<td><img src="image3" alt="PQM_ordinary_diagram" /></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\Sigma$ - curvilinear quadric</td>
</tr>
<tr>
<td>PQM generalized</td>
<td>1</td>
<td>3</td>
<td><img src="image4" alt="PQM_generalized_diagram" /></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\Sigma$ - bundle surface</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\Sigma$ - warped quadric</td>
</tr>
</tbody>
</table>
Table 2. Structure of apparatuses of the PQ mapping projections of $P_4$ space

<table>
<thead>
<tr>
<th>Type of the PQ projection</th>
<th>dim $C_1$</th>
<th>dim $F_1$</th>
<th>Ideological schema of a structure of a projection apparatus</th>
</tr>
</thead>
<tbody>
<tr>
<td>PQJ ordinary</td>
<td>-1</td>
<td>1</td>
<td>$\Sigma$ - envelope conic</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\Sigma$ - warped quadric</td>
</tr>
<tr>
<td>PQJ ordinary</td>
<td>-1</td>
<td>2</td>
<td>$\Sigma$ - envelope quadric</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\Sigma$ - warped hyper quadric</td>
</tr>
<tr>
<td>PQJ generalized</td>
<td>0</td>
<td>2</td>
<td>$\Sigma$ - envelope bundle surface</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\Sigma$ - bundle hyper surface with plane formers</td>
</tr>
<tr>
<td>PQM ordinary</td>
<td>1</td>
<td>3</td>
<td>$\Sigma$ - bundle surface</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\Sigma$ - warped quadric</td>
</tr>
</tbody>
</table>
It is also worth noting that like with subspace projections with bundle dispersed centres, some of the PQ-type projections under consideration are minimal reversible representations [4] of the $P_n$ space. It means that for every point $X_i$ belonging to the $P_n$ space, which is generally situated versus the apparatus of the PQ projection, it is possible to reconstruct the position of this point towards the previously rebuilt apparatus of the PQ projection, using the $X_i^S$ projection as a basis. For example, these are representations with such properties in the PQ family of graphical projections of the $P_3$ and $P_4$ spaces:

- an ordinary PQM projection of the $P_3$ space with its apparatus described in the position 2 in table 1,
- an ordinary PQM projection of the $P_3$ space with its apparatus described in the position 3 in table 1,
- an ordinary PQM projection of the $P_4$ space with its apparatus described in the position 5 in table 2.

Generally speaking, it is possible to prove that the PQ subspace projection of the $P_n$ space onto the projection subspace $P$ is a minimal reversible representation of this space, when it belongs to the PQM-type family, and elements of its apparatus have the following properties:

1° $S_1 \cap K \neq P_n$,
2° $S_1 \cap K = \emptyset$,
3° $\dim S_1 < \frac{1}{2}(\dim F_1 - 1)$.
References


RZUTOWANIA PODPRZESTRZENIOWE PRZESTRZENI RZutowych RZUTOWYCH REALIZOWANE ZE ŚRODKÓW ROZPROSZONYCH NA UTWORACH DRUGIEGO STOPNIA

Zasady działania oraz właściwości rzutowań podprzestrzeniowych, tzn. rzutowań, których środki i utwory rzutowające poszczególne punkty odwzorowywanej przestrzeni są podprzestrzeniami, zostały już wcześniej spręczowane w stosunku do tzw. rzutowań wiązkowych (ze stałego środka) oraz dla podprzestrzeniowych rzutowań ze środków rozproszonych wiązkowo. W niniejszym opracowaniu podaje się ogólne informacje o właściwościach rzutowań podprzestrzeniowych przestrzeni rzutowych realizowanych ze środków rozproszonych na utworach drugiego stopnia.