ON REDUCIBILITY OF THE INTERSECTION CURVE OF TWO SECOND-ORDER SURFACES

Andrzej BIELIŃSKI, Cecylia ŁAPIŃSKA

Warsaw University of Technology, Institute of Heating and Ventilation
20 Nowowiejska st., 00-653 Warsaw, Poland
e-mail: cecylia.lapinska@is.pw.edu.pl

Abstract. A generalization of the well known theorem about the division of the common curve of two quadrics in two parts which are tangent to a common sphere is given.

Key Words: conic, quadric, pole, polar, conjugate lines with respect to a quadric, perspective collineation, elation, geometric homology, harmonic involution.

1. Introduction

Almost each descriptive geometry manual which includes a chapter on second-order surfaces (also called quadratic surfaces or quadrics) gives the theorem about the division in two conics of the common curve of two quadrics which are tangent in two points. Usually a simple proof based on the Pascal’s Theorem is also quoted (see for ex. [1]).

In exercises this basic theorem is often used, in particular in the case when two circular surfaces are circumscribed on the same sphere and their circles of tangency intersect in two points. If the circles of tangency have at most one common point the assumption of the above theorem is not satisfied and thus it cannot be applied there. However, even in this case the thesis of the theorem is true but there is no proof of that fact, which has been observed at the seminar in the Department of Descriptive Geometry of Warsaw Technical University. In the manual by E. Otto ([2], p.209) one can only find an exercise asking to prove that the intersection curve of two cones circumscribed on the same sphere consists of two conics also in the case when the line joining the vertices of the cones has at least one common point with the sphere. The author gives a hint stating that the sections of the cones by an appropriate plane coincide since they have common axis and foci.

In our paper we present a more general theorem which implies the reducibility of the intersection curve of two quadrics circumscribed on a sphere in the general case.

In the further part of the work definitions and terminology are conform to those included in the manual [2]. Unless the contrary is stated, we understand that we are dealing with nonsingular conics and quadrics.

2. Perspective Collineations from Sections of a Quadric onto Sections of a Cone Tangent to the Quadric

Let \( \Phi \) be a quadric. Consider a conic \( s \) obtained by intersecting \( \Phi \) with a plane \( \omega \), the point \( W \) which is the pole of the plane \( \omega \) with respect to the quadric \( \Phi \) and the cone \( \Delta \) with vertex \( W \) tangent to \( \Phi \).

Observe that for every \( P \in \omega \) the polar plane \( \pi_\Delta \) of \( P \) with respect to \( \Delta \) and the polar plane \( \pi_\Phi \) of \( P \) with respect to \( \Phi \) coincide because each of them contains the pole \( W \) of \( \omega \) and the polar line \( p \) of \( P \) with respect to the conic \( s \). Therefore every line \( \tilde{l} \) conjugate to a line \( l \) lying in \( \omega \) with respect to \( \Phi \) is also conjugate to \( l \) with respect to \( \Delta \).
Let now a plane \(\alpha\) intersect the quadric \(\Phi\) in a conic \(s_\Phi\) and the cone \(\Delta\) in a conic \(s_\Delta\) (Figure 1). Denote by \(k\) the edge of the planes \(\alpha\) and \(\omega\) (\(k = \alpha \cap \omega\)). The line \(\bar{k}\) which is conjugate to \(k\) with respect to \(\Phi\) and \(\Delta\) cuts through the plane \(\alpha\) in the point \(K\) (t.i. \(K = \alpha \cap \bar{k}\)).

With the above assumptions and notations we have the following statements.

**Lemma 1:** If the conic \(s_\Phi\) is nonsingular, then there exists a perspective collineation which maps \(s_\Phi\) onto \(s_\Delta\). The point \(K\) is the center and the line \(k\) the axis of the collineation.

**Proof:** It is known (see for ex. [2] p.163) that for any two nonsingular conics lying on a quadric there is a perspective collineation of three-dimensional space which transforms one of them into the other and the quadric onto itself. So we have a perspective collineation \(f : X \mapsto X'\) which maps \(\Phi\) onto \(\Phi\), the conic \(s_\Phi\) onto \(s\) and the plane \(\alpha\) onto the plane \(\omega\). Let \(\gamma\) be the fundamental plane of the collineation, t.i. \(f(X) = X\) for every \(X \in \gamma\). Evidently \(k \subset \gamma\), so \(k\) is invariant, \(f(k) = k\). Since perspective collineations preserve conjugate elements, we have also \(f(k) = \bar{k}\).

![Figure 1](image)

Take now a perspective collineation \(g : X' \mapsto X''\) mapping \(\omega\) onto \(\alpha\) with the centre \(W\) and the fundamental plane \(\gamma\) considered above. Clearly, the cone \(\Delta\) is mapping onto itself and the conic \(s\) is transformed into the conic section \(s_\Delta\). We have also \(g(\bar{k}) = \bar{k}\).

As products of two perspective collineations with the same fundamental plane are perspective collineations, we can state that the product \(h\) of \(f\) and \(g\) is a perspective collineation, \(X'' = h(X) = g(f(X))\). The image of the plane \(\alpha\) under \(h\) is the plane \(\alpha\). We have also \(h(\bar{k}) = g(g(\bar{k})) = g(\bar{k}) = \bar{k}\). Hence, the point \(K = \alpha \cap \bar{k}\) is not changed under \(h\) and it is the centre of \(h\). Now we can define a perspective collineation from \(\alpha\) to \(\alpha\) as the restriction of \(h\) to the
plane \( \alpha \) whose centre is \( K \) and axis is the line \( k \). Of course, the image of the conic \( s_\Phi \) under this collineation is \( s_\Phi \). The proof is complete.

Remark that the point \( K \) is the pole of the line \( k \) with respect to both \( s_\Phi \) and \( s_s \).

**Lemma 2:** Let two quadrics \( \Phi_1 \) and \( \Phi \) be tangent and let \( s \) be the conic of contact. If a plane \( \alpha \) intersects the quadrics \( \Phi_1 \) and \( \Phi \) in the conics \( s_1 \) and \( s_\Phi \) respectively, then there exists a perspective collineation mapping \( s_1 \) onto \( s_\Phi \).

**Proof:** The quadrics \( \Phi_1 \) and \( \Phi \) have a common cone \( \Delta \) tangent to them along the conic \( s \). Denote by \( s_1 \) the section of \( \Delta \) by the plane \( \alpha \). By Lemma 1 there exists a perspective collineation \( h_1 \) mapping \( s_1 \) onto \( s_\Phi \) and a perspective collineation \( h \) mapping \( s_\Phi \) onto \( s_\Phi \). The point \( K \) and the line \( k \) are the common center and the common axis of the collineations \( h_1 \) and \( h \). Hence the product \( f \) of the collineations \( h_1 \) and the inverse \( h^{-1} \) of \( h \) is a perspective collineation with the centre \( K \) and axis \( k \) which maps \( s_1 \) onto \( s_\Phi \).

**Lemma 3:** Let \( \Phi_1, \Phi_2 \) and \( \Phi \) be quadrics. Let \( \Phi_1 \) and \( \Phi_2 \) be tangent to \( \Phi \) and their curves of contact lie in the planes \( \alpha_1 \) and \( \alpha_2 \) respectively. If a plane \( \alpha \) including the edge of the planes \( \alpha_1 \) and \( \alpha_2 \) intersects the quadrics \( \Phi_1 \) and \( \Phi_2 \) in nonsingular conics \( s_1 \) and \( s_2 \) respectively, then there exists a perspective collineation which maps \( s_1 \) onto \( s_2 \).

**Proof:** Consider the line \( k = \alpha_1 \cap \alpha_2 \). The line \( \overline{k} \) conjugate to \( k \) with respect to \( \Phi \) is also conjugate to \( k \) with respect to \( \Phi_1 \) and \( \Phi_2 \). Hence both the collineations \( f_1 \) and \( f_2 \) which exist by Lemma 2 ( \( f_1 \) mapping \( s_1 \) onto \( s_\Phi \) and \( f_2 \) mapping \( s_2 \) onto \( s_\Phi \) ) have the same centre \( K \) and the same axis \( k \). The product of \( f_1 \) and the inverse of \( f_2 \) is a perspective collineation with the axis \( k \) and centre \( K \) and it maps \( s_1 \) onto \( s_2 \).

By an analogous reasoning we can also obtain the following lemma.

**Lemma 4:** If quadrics \( \Phi_1 \) and \( \Phi_2 \) are tangent to the same cone \( \Delta \) with the conics of contact lying on the planes \( \alpha_1 \) and \( \alpha_2 \) respectively, then for every plane \( \alpha \) including the edge of the planes \( \alpha_1 \) and \( \alpha_2 \) which intersects the quadrics \( \Phi_1 \) and \( \Phi_2 \) in nonsingular conics \( s_1 \) and \( s_2 \) respectively, there exists a perspective collineation which maps \( s_1 \) onto \( s_2 \).

**3. Reducibility of Intersection Curve of Two Second-Order Non-degenerate Surfaces**

**Lemma 5:** Let \( \Phi_1, \Phi_2 \) and \( \Phi \) be quadrics such that \( \Phi_1 \) is tangent to \( \Phi \) and their conic of contact lies on a plane \( \alpha_1 \), \( \Phi_2 \) is tangent to \( \Phi \) and their conic of contact lies on a plane \( \alpha_2 \). If there exists a common point \( P \) of the quadrics \( \Phi_1 \) and \( \Phi_2 \) which does not lie on the edge \( k \) of the planes \( \alpha_1 \) and \( \alpha_2 \) then the sections \( s_1 \) and \( s_2 \) of \( \Phi_1 \) and \( \Phi_2 \) respectively by the plane \( \alpha \) including the point \( P \) and the line \( k = \alpha_1 \cap \alpha_2 \) coincide if they are nonsingular.

**Proof:** By Lemma 3 there exists a perspective collineation \( f \) with the \( k = \alpha_1 \cap \alpha_2 \) and the centre \( K = \overline{k} \cap \alpha \) which maps \( s_1 \) onto \( s_2 \). The point \( P \) belongs to \( s_1 \) and \( s_2 \) and \( P \notin k \) and \( P \notin K \). If \( f \) is an elation we have necessarily \( f(P) = P \). Hence in this case \( f \) is an identity and \( s_1 = s_2 \). If \( f \) is a geometric homology we cannot exclude that \( f(P) = P \neq P \). But \( K \) being the pole and \( k \) its polar with respect to \( s_2 \), in this case because \( f(P) \in s_2 \) the collineation \( f \) interchanges the points \( P \) and \( P_1 \), so it is an involution on \( s_2 \). As \( f(s_1) = s_2 \) and \( f \) is one-to-one we have necessarily \( s_1 = s_2 \), which completes the proof.

Let us remark now that the intersection curve of two second-order algebraic surfaces is a four-order curve. Hence if two quadrics have a common conic then the common points of these quadrics which does not belong to this conic form a second order curve or a conic. If the assumptions of Lemma 4 are satisfied the quadrics \( \Phi_1 \) and \( \Phi_2 \) have a common nonsingular conic \( s_1 = s_2 \) whose plane passes through the line \( k = \alpha_1 \cap \alpha_2 \). Thus the intersection curve of the quadrics \( \Phi_1 \) and \( \Phi_2 \) is reduced to two conics.

We summarise the preceding remarks in the following theorem.

**Theorem 1:** If quadrics \( \Phi_1 \) and \( \Phi_2 \) are tangent to a quadric \( \Phi \) along conics \( s_1 \) and \( s_2 \) respectively, \( s_1 \) lying on the plane \( \alpha_1 \) and \( s_2 \) on the plane \( \alpha_2 \), and have a common point \( P \), such
that \( P \not\in a_1 \cap a_2 \), then the intersection curve of the quadrics \( \Phi_1 \) and \( \Phi_2 \) consists of two conics\(^1\) whose planes pass through the edge of the planes of the conics of contact.

**Corollary 1**: If two quadrics are circumscribed on the same sphere then their intersection curve divides into two conics.

Observe now that the surfaces \( \Phi_1, \Phi_2 \) and \( \Phi \) in Theorem 1 are not necessarily all non-singular quadrics. Two changes in the assumptions are possible. In fact, by applying Lemma 3 it is not difficult to prove, by the same argumentation as that in the proof of Lemma 5, that if the assumptions about \( \Phi_1 \) and \( \Phi_2 \) remain unchanged the thesis of Theorem 1 holds even when \( \Phi \) is a cone. On the other hand, by examining the argumentation of the proof of Theorem 1 and those of the preceding lemmas it is easy to see that if \( \Phi \) is a cone the thesis of Theorem 1 is valid although we replace one or both of \( \Phi_1 \) and \( \Phi_2 \) by cones.

The above observations can be summarised in the following statements.

**Theorem 2**: If quadrics \( \Phi_1 \) and \( \Phi_2 \) are tangent to a cone \( \Delta \) along conics \( s_1 \) and \( s_2 \) respectively, \( s_1 \) lying on the plane \( a_1 \) and \( s_2 \) lying on the plane \( a_2 \), and have a common point \( P \), \( P \not\in a_1 \cap a_2 \), then the intersection curve of the quadrics \( \Phi_1 \) and \( \Phi_2 \) consists of two conics\(^1\) whose planes pass through the edge \( k \) of the planes of the conics of contact, \( k = a_1 \cap a_2 \).

**Corollary 2**: If two quadrics inscribed in the same cone are not disjoint their intersection curve consists of two conics\(^1\).

**Theorem 3**: Let two second-ordered non-degenerated surfaces \( S_1 \) and \( S_2 \) be tangent to a quadric \( \Phi \) along the conics of contact \( s_1 \) and \( s_2 \) lying on the planes \( a_1 \) and \( a_2 \) respectively. If there exists a common point \( P \), \( P \not\in a_1 \cap a_2 \), then the intersection curve of the surfaces \( S_1 \) and \( S_2 \) consists of two conics\(^1\) whose planes pass through the edge \( k \) of the planes of the conics of contact.

**Corollary 3**: Every two polar cones (i.e. consisting of tangents to quadric) of the same quadric intersect along conics\(^1\).

---

\(^1\) may be singular
3. Example

We give a simple example to illustrate our considerations.

Figure 2 presents a projection on the symmetry plane of the curve of intersection of two cones $\Delta_1$ and $\Delta_2$ circumscribed on the same sphere. The intersection curve of these cones reduces to an ellipse and a hyperbola whose planes pass through the edge $k$ of the planes $\alpha_1$ and $\alpha_2$ including the circles of contact of the cones and the sphere.

References

O ROZPADZIE LINII PRZENIKANIA POWIERZCHNI DRUGIEGO STOPNIA

W pracy przedstawiono dowód twierdzenia o rozpadzie linii przenikania dwóch powierzchni drugiego stopnia stycznych do wspólnej kwadryki wzdłuż stożkowych. Idea dowodu polega na ustaleniu kolineacji środków zachodzących pomiędzy płaszczyznami stożkowych styczności i dowolną płaszczyzną, a następnie, korzystając z kolineacji pomiędzy przekrojami przenikającymi się powierzchni odpowiednio dobraną płaszczyzną, pokazanie, że przekroje te jednoczą się, uzyskując w ten sposób wspólną stożkową obu powierzchni. Sformułowano i udowodniono analogiczne twierdzenie dla dwóch kwadryk wpisanych w ten sam stożek.

Reviewer: Prof. Bogusław GROCHOWSKI, DSc

Received December 20, 2004