CONSTRUCTION OF HYPERBOLA

Stanisław OCHOŃSKI

Retired academic teacher of Częstochowa University of Technology, Faculty of Building,
Department of Descriptive Geometry and Engineering Graphics
Lednica Górna 16, 32-020 Wieliczka, Poland
email: stanislawo@o2.pl

Abstract: In this article, the author gives an interesting and relatively simple construction of a hyperbola which is determined by its asymptotes and a random point. In the case of equilateral hyperbola this construction can also be implemented when the hyperbola is given with an imaginary axis and the vertex, and in the general case by an imaginary axis, the center of concentric circles and a random real number n>0. The proposed method can also be implemented in the case of the hyperbola given by its vertices and a point. The construction of the subsequent points of hyperbola was deduced from the properties of straight line transformation (as degenerate of a conic) by means of the pencil of concentric circles.

Keywords: equilateral hyperbola, asymptotes, pencil of concentric circles

1 Introduction

In the work [1] the transformation of pairs of straight lines (parallel/intersecting) by means of a pencil of concentric circles implemented for the construction of equilateral triangles whose vertices belong, respectively, to three given lines. And in the paper [2] from thus determined transformation of a circle an original and relatively simple construction of a parabola was deduced with the following elements: a vertex, an axis and a random point.

In this paper the author subsequently proposes an original and relatively simple construction of a hyperbola of successive points inferred from the properties of straight line (as a degenerate conic) transformation with a pencil of concentric circles.

The first part of paper contains the definition of a transformation and the proof of a theorem declaring that the image of the straight line in this transformation is a pencil of concentric and coaxial hyperbolas (for which the given straight line is an imaginary axis).

The second part of this article gives the algorithms of construction of successive points of the hyperbola determined by the following elements: a) asymptotes and any point, b) imaginary axis, the center of concentric circles and any real number n>0 and n≠∞, c) vertices and a random point, and in the case of the equilateral hyperbola, d) an imaginary axis and vertex.

2 Transformation of straight line

Let us assume a degenerated conic reduced to a double straight line a=b and also the point O (not belonging to it) on the plane of drawing, as the center of the pencil of concentric circles. Let us assume that this line is the axis of ordinate y of the orthogonal coordinate system whose axis of abscissa x includes point O (Fig.1) In thus determined reference system M(x,y), the equations of the straight line a=b=y and the circles $o_i(O,R_i)$ are in the shape of:

$$x = 0 \text{ and} \quad (1)$$

$$(x - e)^2 + y^2 = R_i^2 \quad (2)$$

ISSN 1644-9363 / PLN 15.00

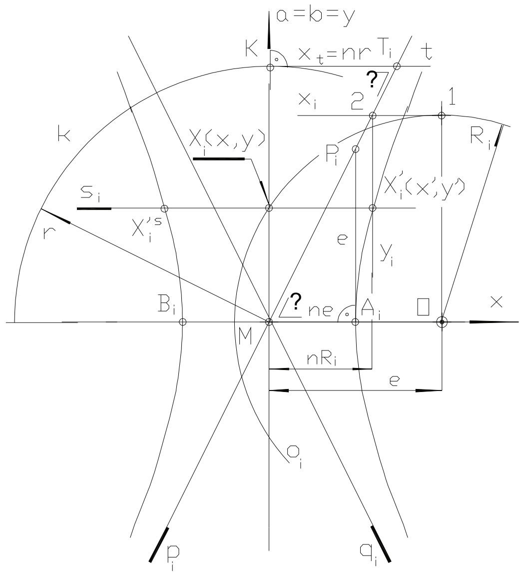

Figure 1

The common points $X_i(x,y)$ (as well as those symmetrical to them with respect to axis **x**) of the straight line (1) and the circles (2) were assigned the points X_i',$X_i'^s(x',y')$, respectively belonging to the lines s_i parallel to the axis x, whose ordinates y'=y (in the same system of reference). The abscissa x' of these points can be equal to the arbitrary values of radii R_i.

These determinations result in the following formulas of this transformation:

$$x' = n\cdot R_i \ (0 \le n \le \infty) \text{ and } y' = y \tag{3}$$

By introducing the relation (3) into the equation (2) instead of y and R_i, we obtain:

$$(x - e)^2 = x'^2/n^2 - y'^2 \text{ , and after rearrangement:} \tag{4}$$

$$x = e \pm \sqrt{\frac{x'^2}{n^2} - y'^2} \text{ , and because x=0, we have:} \tag{5}$$

$$e \pm \sqrt{\frac{x'^2}{n^2} - y'^2} = 0 \text{ , and after transformation we obtain:} \tag{6}$$

$$\frac{x'^2}{n^2e^2} - \frac{y'^2}{e^2} = 1 \text{ – it is the equation of pencil of hyperbolas} \tag{7}$$

From the equation (7) for y' = 0 we receive the abscissa $x' = \pm n\cdot e$ of the vertices A_i, B_i, and for n = 1, the equation (7) takes shape:

$$\frac{x'^2}{e^2} - \frac{y'^2}{e^2} = 1 \text{ – it is the equilateral hyperbola} \tag{8}$$

If n = 0 then hyperbola is reduced to the straight line a=b=y and while n = ∞ – to the axis of abscissa x.

It is the analytic proof of the statement declaring that the image of a straight line (as a degenerate conic) in thus determined transformation is a pencil of concentric and coaxial hyperbolas. Knowing the vertices of the hyperbola and the lengths of its axes we can construct its asymptotes ($tg\alpha = \frac{e}{ne}$).

It can be pointed out that these asymptotes can also be constructed in the following way. The semicircle k with the center M and the random radius r intersects axis y at point K.

Then tangent t at this point to the circle k intersects the line p_i at point T_i. The segment KT_i is the abscissa x_t of the point T_i. From the right – angled triangles MA_iP_i and MKT_i it follows that:

$$tg\alpha = \frac{e}{ne} = \frac{r}{x_t} \text{ and hence we receive:} \qquad (9)$$

$$x_t = nr. \qquad (10)$$

The straight line p_i joining the point T_i with the abscissa $x_t = nr$ with the point M is the asymptote of the hyperbola.

The relation (10) was used, in the point 3, for the graphical method of construction of successive points of the hyperbola.

These considerations result in the following conclusion - that non-singular hyperbola can be determined by its imaginary axis, the center of concentric circles which does not belong to them and a random real number $n>0$ and $n\neq\infty$.

3 Construction of hyperbola

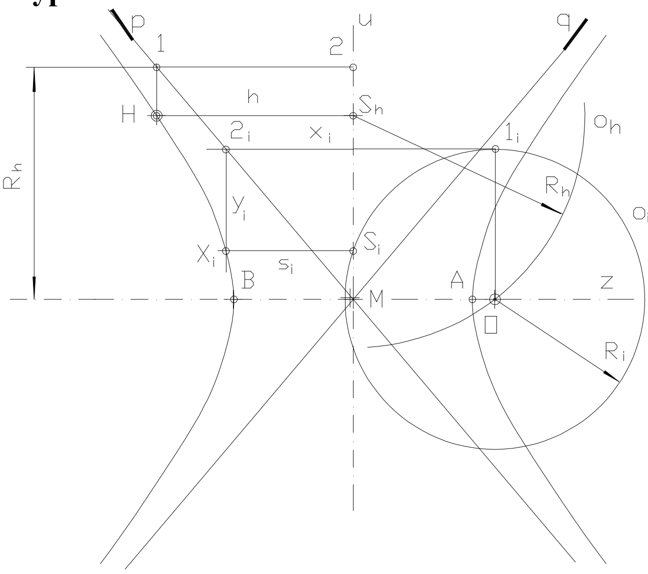

Figure 2

Using the properties of transformation of the straight line (as a regular conic) with a pencil of concentric circles and the statement proved in the point 2 and also the deduced corollary now we can give the algorithms of construction of the successive points of the hyperbola determined by its given and indispensable elements. The first algorithm gives the construction of the next points of hyperbola determined by its asymptotes and a random point (Fig.2). The second algorithm presents the construction of successive points of the equilateral hyperbola determined by its asymptotes and a random point (Fig.3). In the third algorithm the equilateral hyperbola is given with its imaginary axis and a vertex (Fig.4). Finally, in the fourth algorithm the hyperbola is defined (due to the corollary) by its imaginary axis, the center of concentric circles and any real number $n>0$ and $n\neq\infty$ (Fig.5).

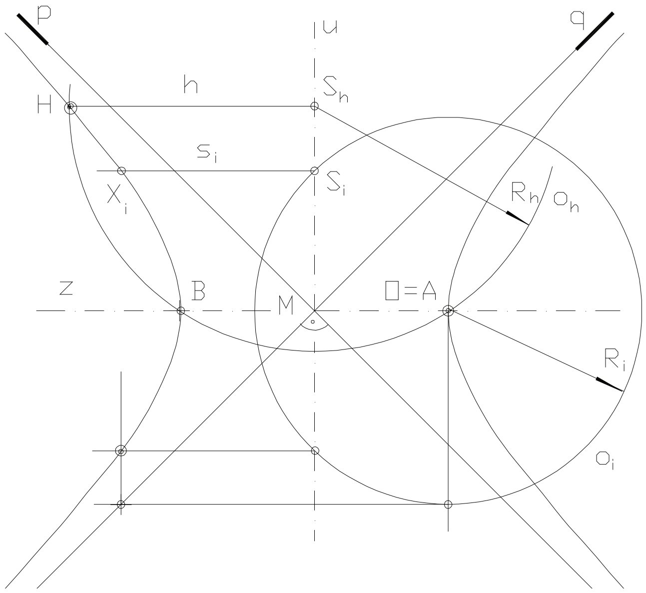

Figure 3

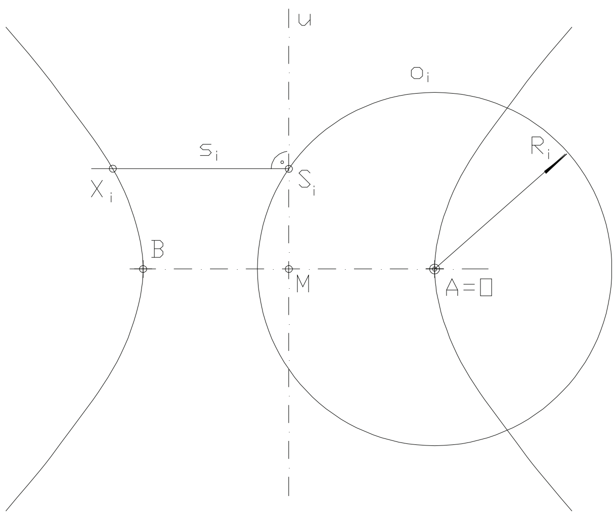

Figure 4

Algorithm 1 (given: p and q and also the point H)

- we trace the axes of hyperbola z and u;
- we have $H \in h \| z$; $h \cap u = \{S_h\}$; $H1 \| u$; $12 \| z$; $|M2| = R_h$;
- $o_h(S_h,R_h) \cap z = \{O\}$ is the center of concentric circles;
- $o_i(O,R_i) \cap u = \{S_i\}$; $S_i \in s_i \| z$; $x_i \| z$ and tangent to circle $o_i(O,R_i)$;
- $x_i \cap p = \{2_i\}$; $2_i \in y_i \| u$; $s_i \cap y_i = \{X_i\}$– a point of the hyperbola (Fig.2).

Algorithm 2 (given: $p \perp q$ and the point H)

- we trace the axes of hyperbola z and u;
- $H \in h \| z$; $h \cap u = \{S_h\}$;
- $o_h(S_h,R_h=|S_hH|) \cap z = \{A, B\}$, O=A (or O=B) is the center of concentric circles and at once the vertex of hyperbola;

- $o_i(O,R_i) \cap u = \{S_i\}$; $S_i \in s_i \parallel z$; $|S_iX_i| = R_i$; X_i is a point of the hyperbola (Fig.3).

Algorithm 3 (given: an imaginary axis u and a vertex A)

- A = O is the center of concentric circles;
- $o_i(O,R_i) \cap u = \{S_i\}$; $S_i \in s_i \perp u$; $|S_iX_i| = R_i$; X_i is a point of the hyperbola (Fig.4).

Algorithm 4 (given: an imaginary axis u, the center O and $n = \frac{1}{3}$)

- $o_i(O,R_i) \cap u = \{S_i\}$; $S_i \in s_i \perp u$; $X_i \in s_i$ and $|S_iX_i| = \frac{1}{3} R_i$;
- X_i is a point of the hyperbola (Fig.5).

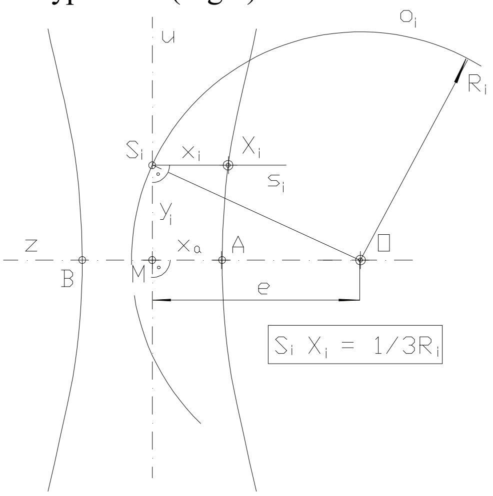

Figure 5

Let us still consider the possibility of application of the method described above, of construction of the successive points of the hyperbola determined by its vertices A and B and also a random point X_i (Fig.5). Perpendicular bisector of the segment AB is its imaginary axis and at once the axis of the ordinate y of the orthogonal system of reference M(x,y), whose axis of abscissa x includes the vertices (Fig.1).

In the rectangular triangle S_iMO (Fig.5) we have

$R_i^2 = e^2 + y_i^2$, where $e = |MO|$, and y_i is the ordinate of the given point X_i. (11)

From the relation (10) we have

$e = \frac{x_a}{n}$ and $R_i = \frac{x_i}{n}$, where x_a and x_i are the abscissas of the given points A and X_i, respectively. (12)

Introducing the relation (12) into the equation (11) instead of e and R_i, after transformation we obtain:

$$n = \sqrt{\frac{x_i^2 - x_a^2}{y_i}} \quad \text{(given: } x_i, x_a \text{ and } y_i \text{ and } x_i > x_a\text{)}. \qquad (13)$$

For example, for $x_i = 5$, $x_a = 3$ and $y_i = 4$, n = 1 (equilateral hyperbola, O=A or O=B i.e. e=3) and for y_i=8, $n=\frac{1}{2}$, e=6 and for y_i=12, $n=\frac{1}{3}$, e=9. Whereas for x_i=2.2, x_a=1.3 and y_i=2.5, n=0.71 and e=1.8.

Knowing an imaginary axis of the hyperbola, the center of concentric circles, if n=1, then we realize the construction of successive points of this hyperbola according to the third algorithm, and while n≠1 we follow the procedures of the fourth algorithm.

References :

[1] Ochoński S.: *The Equilateral Triangles whose Vertices Belong to Three Non Coplanar Straight Lines*. The Journal of Polish Society for Geometry and Engineering Graphics, Volume 19 (2009), 15-26.

[2] Ochoński S.: *Construction of Parabola*. The Journal of Polish Society for Geometry and Engineering Graphics, Volume 20 (2009), 13-16.

KONSTRUKCJA HIPERBOLI

W prezentowanym artykule podano oryginalną i stosunkowo prostą konstrukcję hiperboli określonej jej niezbędnymi elementami. Konstrukcję kolejnych punktów hiperboli wyprowadzono z właściwości przekształcenia prostej, jako stożkowej zdegenerowanej, za pomocą pęku koncentrycznych okręgów.

Pierwsza część artykułu zawiera definicję przekształcenia, analityczny dowód twierdzenia orzekającego, iż obrazem prostej w tym przekształceniu jest pęk współśrodkowych i współosiowych hiperbol, dla którego przekształcana prosta jest wspólną osią urojoną oraz wyprowadzony wniosek, że hiperbola może być również określona osią urojoną, środkiem koncentrycznych okręgów oraz dowolną liczą $n > 0$ i $n \neq \infty$.

W drugiej części pracy podano algorytmy konstrukcji bieżących punktów hiperboli zadanej: a) jej asymptotami i dowolnym punktem, b) osią urojoną, środkiem koncentrycznych okręgów i dowolną liczbą n, c) wierzchołkami i dowolnym punktem, a w przypadku hiperboli równobocznej d) osią urojoną i wierzchołkiem.