A GROUP OF SPHERICAL TESSELLATIONS BASED ON SECOND FAMILY REGULAR OCTAHEDRON-BASED POLYHEDRA

Dominika BYSIEC

Kielce University of Technology, 1000-lecia PP ave., 7, 25-314 Kielce, POLAND

Abstract: The aim of this study was to calculate geometric parameters of octahedron-based polyhedra generated according to the second method of subdivision each equilateral face into smaller triangular elements. The analysis was generated for 6 derivative polyhedra and on the basis of them the final conclusions were prepared. Some formulas were invented and algorithms were developed using MsExcel program.

Keywords: octahedron-based polyhedra, division of equilateral triangle, angular coordinates, rectangular coordinates, area, volume, edge length.

Introduction

To generate octahedron (Platonic)-based polyhedra [1], each of eight congruent equilateral triangles which are faces – were divided according to one of the three methods (Fig. 1) [2].

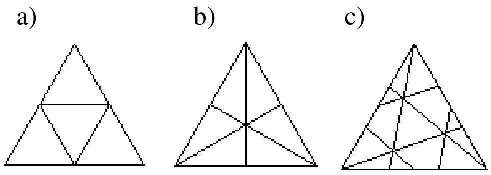

Fig.1. Three methods of triangular faces division.

Octahedron-based polyhedra with greater number of faces were obtained on the basis of this division and tabulated in Table 1 [3] [4]. Main diagonal of table presents octahedron-based polyhedra obtained by applying first method of equilateral triangle division (32-hedron, 72-hedron, 128-hedron, etc.). Polyhedra located on bisector of the upper part of table present second method of division (24-hedron, 96-hedron, 216-hedron, etc.). The third method of division is shown by polyhedra located above and below bisector (56-hedron, 104-hedron, 152-hedron, etc.).

Table.1. Octahedron-based polyhedra; t,k – consecutive natural numbers.

t \ k	1	2	3	4	5	6	7	8	...
1	8	**24**	56	104	168	248	344	456	
2	24	32	56	**96**	152	224	312	416	
3	56	56	72	104	152	**216**	296	392	
4	104	96	104	128	168	224	296	**384**	
5	168	152	152	168	200	248	312	392	
6	248	224	216	224	248	288	344	416	
7	344					344	392	486	
8								512	
...									...

ISSN 1644-9363 / PLN 15.00

The second method of equilateral triangle division was analyzed in this study (Fig.1b), consisting in each its face division into **n** parts and drawing three parallel line groups through division points to height of each edge (Fig.2).

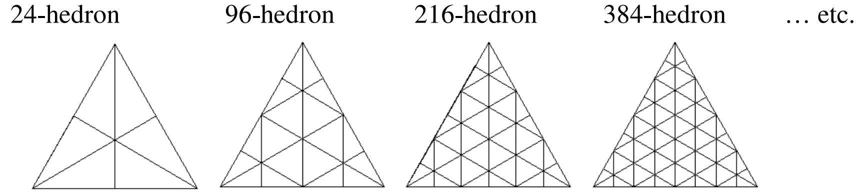

Fig.2. One face of further octahedron-based polyhedra using the second method of equilateral triangle division

Spherical and rectangular coordinates calculation

The flat grids were generated**:** a semi-central view of one half of the grid of a 864-hedron (Fig.3) and one face of octahedron and its derivatives (Fig.4) where all vertices were marked.

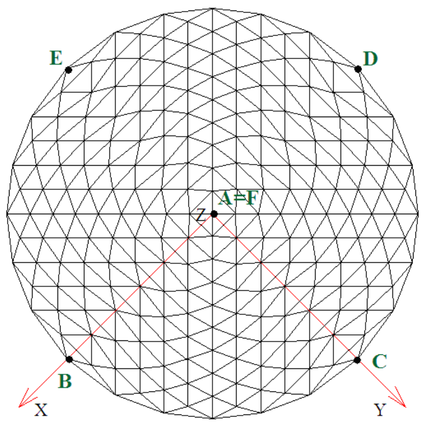

Fig.3. A semi-central view of one half of the grid of a 864-hedron

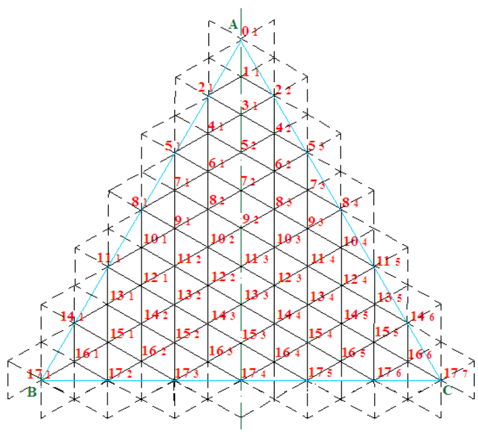

Fig.4. The flat grid of 1/8 part of a 864-hedron with all vertices marked

The geometric parameters of octahedron-based polyhedra were calculated: angular co-ordinates, rectangular coordinates, edge lengths, areas and volumes.

The calculations were made for assumed angular and rectangular coordinate system (Fig.5).

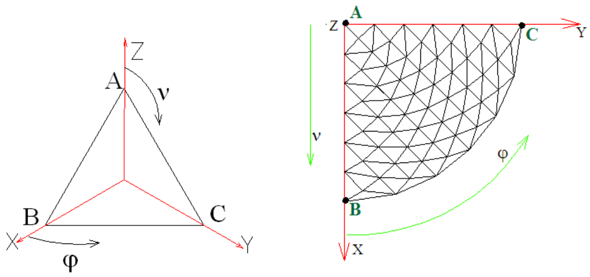

Fig.5. Coordinate system assumed

The angular coordinates results are shown in Table 2 and 3. The symbols used in the tables are: **n** – consecutive positions of the vertices of the generated polyhedra (layers of the edge division), **m** – consecutive rows of the edge division in the original octahedron.

Table.2. Angular coordinates φ of octahedron-based polyhedra

angular coordinates φ [º]									
m \ n	1	2	3	4	5	6	7	…	n
1	45								
2	0	90							
3	45								
4	18	72							
5	0	45	90						
6	25,714285	64,285713							
7	11,25	45	78,75						
8	0	30	60	90					
9	18	45	72						
10	8,181818	32,727272	57,272726	81,818180					
11	0	22,5	45	67,5	90				
12	13,846153	34,615383	55,384613	76,153843					
13	6,428571	25,714285	45	64,285714	83,571428				
14	0	18	36	54	72	90			
15	11,25	28,125	45	61,875	78,75				
16	5,294117	21,176469	37,058821	52,941173	68,823525	84,705877			
17	0	15	30	45	60	75	90		
…									
m									…

Table.3. Angular coordinates υ of octahedron-based polyhedra

angular coordinates υ [º]							
m	24-hedron	96-hedron	216-hedron	384-hedron	600-hedron	864-hedron	…-hedron
1	60	30	20	15	12	10	
2	90	45	30	22,5	18	15	
3		60	40	30	24	20	
4		75	50	37,5	30	25	
5		90	60	45	36	30	
6			70	52,5	42	35	
7			80	60	48	40	
8			90	67,5	54	45	
9				75	60	50	
10				82,5	66	55	
11				90	72	60	
12					78	65	
13					84	70	
14					90	75	
15						80	
16						85	
17						90	
…							
m							

ISSN 1644-9363 / PLN 15.00

Basing on the formulas (1) and angular coordinates results, rectangular coordinates of octahedron-based polyhedra were calculated and tabulated in Table 4.

$$X = R * \cos\varphi * \sin\upsilon$$
$$Y = R * \sin\varphi * \sin\upsilon \qquad (1)$$
$$Z = R * \cos\upsilon$$

Table.4. Rectangular coordinates X, Y, Z of octahedron-based polyhedra

	24-hedron			96-hedron			216-hedron		
vertices	X	Y	Z	X	Y	Z	X	Y	Z
01	0,000000	0,000000	1,000000	0,000000	0,000000	1,000000	0,000000	0,000000	1,000000
11	0,612372	0,612372	0,500000	0,353554	0,353554	0,866025	0,241845	0,241845	0,939693
21	1,000000	0,000000	0,000000	0,707107	0,000000	0,707107	0,500000	0,000000	0,866025
22	0,000000	1,000000	0,000000	0,000000	0,707107	0,707107	0,000000	0,500000	0,866025
31				0,612372	0,612372	0,500000	0,454520	0,454520	0,766044
41				0,918651	0,298488	0,258819	0,728552	0,236721	0,642788
42				0,298488	0,918651	0,258819	0,236721	0,728552	0,642788
51				1,000000	0,000000	0,000000	0,866025	0,000000	0,500000
52				0,707107	0,707107	0,000000	0,612372	0,612372	0,500000
53				0,000000	1,000000	0,000000	0,000000	0,866025	0,500000
61							0,846634	0,407718	0,342020
62							0,407718	0,846634	0,342020
71							0,965885	0,192126	0,173648
72							0,696365	0,696365	0,173648
73							0,192126	0,965885	0,173648
81							1,000000	0,000000	0,000000
82							0,866025	0,500000	0,000000
83							0,500000	0,866025	0,000000
84							0,000000	1,000000	0,000000
91									

Table 4. continued

	384-hedron			600-hedron			864-hedron		
vertices	X	Y	Z	X	Y	Z	X	Y	Z
01	0,000000	0,000000	1,000000	0,000000	0,000000	1,000000	0,000000	0,000000	1,000000
11	0,183013	0,183013	0,965926	0,147016	0,147016	0,978148	0,122788	0,122788	0,984808
21	0,382683	0,000000	0,923880	0,309017	0,000000	0,951057	0,258819	0,000000	0,965926
22	0,000000	0,382683	0,923880	0,000000	0,309017	0,951057	0,000000	0,258819	0,965926
31	0,353554	0,353554	0,866025	0,287607	0,287607	0,913545	0,241845	0,241845	0,939693
41	0,578966	0,188117	0,793353	0,475529	0,154509	0,866025	0,401934	0,130596	0,906308
42	0,188117	0,578966	0,793353	0,154509	0,475529	0,866025	0,130596	0,401934	0,906308
51	0,707107	0,000000	0,707107	0,587785	0,000000	0,809017	0,500000	0,000000	0,866025
52	0,500000	0,500000	0,707107	0,415627	0,415627	0,809017	0,353554	0,353554	0,866025
53	0,000000	0,707107	0,707107	0,000000	0,587785	0,809017	0,000000	0,500000	0,866025
61	0,714786	0,344223	0,608761	0,602866	0,290325	0,743145	0,516774	0,248865	0,819152
62	0,344223	0,714786	0,608761	0,290325	0,602866	0,743145	0,248865	0,516774	0,819152
71	0,849384	0,168953	0,500000	0,728865	0,144980	0,669131	0,630437	0,125402	0,766044
72	0,612372	0,612372	0,500000	0,525483	0,525483	0,669131	0,454520	0,454520	0,766044
73	0,168953	0,849384	0,500000	0,144980	0,728865	0,669131	0,125402	0,630437	0,766044
81	0,923880	0,000000	0,382683	0,809017	0,000000	0,587785	0,707107	0,000000	0,707107
82	0,800103	0,461940	0,382683	0,700629	0,404509	0,587785	0,612372	0,353554	0,707107
83	0,461940	0,800103	0,382683	0,404509	0,700629	0,587785	0,353554	0,612372	0,707107
84	0,000000	0,923880	0,382683	0,000000	0,809017	0,587785	0,000000	0,707107	0,707107
91	0,918651	0,298488	0,258819	0,823639	0,267616	0,500000	0,728552	0,236721	0,642788
92	0,683013	0,683013	0,258819	0,612372	0,612372	0,500000	0,541675	0,541675	0,642788
93	0,298488	0,918651	0,258819	0,267616	0,823639	0,500000	0,236721	0,728552	0,642788
10 1	0,981353	0,141097	0,130526	0,904246	0,130011	0,406737	0,810814	0,116578	0,573576
10 2	0,834057	0,536016	0,130526	0,768523	0,493900	0,406737	0,689115	0,442867	0,573576
10 3	0,536016	0,834057	0,130526	0,493900	0,768523	0,406737	0,442867	0,689115	0,573576
10 4	0,141097	0,981353	0,130526	0,130011	0,904246	0,406737	0,116578	0,810814	0,573576
11 1	1,000000	0,000000	0,000000	0,951057	0,000000	0,309017	0,866025	0,000000	0,500000
11 2	0,923880	0,382683	0,000000	0,878663	0,363953	0,309017	0,800103	0,331413	0,500000
11 3	0,707107	0,707107	0,000000	0,672499	0,672499	0,309017	0,612372	0,612372	0,500000
11 4	0,382683	0,923880	0,000000	0,363953	0,878663	0,309017	0,331413	0,800103	0,500000
11 5	0,000000	1,000000	0,000000	0,000000	0,951057	0,309017	0,000000	0,866025	0,500000
12 1				0,949725	0,234086	0,207912	0,879973	0,216894	0,422618

12 2				0,805000	0,555652	0,207912	0,745877	0,514842	0,422618
12 3				0,555652	0,805000	0,207912	0,514842	0,745877	0,422618
12 4				0,234086	0,949725	0,207912	0,216894	0,879973	0,422618
13 1				0,988268	0,111351	0,104528	0,933784	0,105212	0,342020
13 2				0,896033	0,431507	0,104528	0,846634	0,407718	0,342020
13 3				0,703233	0,703233	0,104528	0,664463	0,664463	0,342020
13 4				0,431507	0,896033	0,104528	0,407718	0,846634	0,342020
13 5				0,111351	0,988268	0,104528	0,105212	0,933784	0,342020
14 1				1,000000	0,000000	0,000000	0,965926	0,000000	0,258819
14 2				0,951057	0,309017	0,000000	0,918651	0,298488	0,258819
14 3				0,809017	0,587785	0,000000	0,781451	0,567757	0,258819
14 4				0,587785	0,809017	0,000000	0,567757	0,781451	0,258819
14 5				0,309017	0,951057	0,000000	0,298488	0,918651	0,258819
14 6				0,000000	1,000000	0,000000	0,000000	0,965926	0,258819
15 1							0,965885	0,192126	0,173648
15 2							0,868523	0,464236	0,173648
15 3							0,696365	0,696365	0,173648
15 4							0,464236	0,868523	0,173648
15 5							0,192126	0,965885	0,173648
16 1							0,991945	0,091917	0,087156
16 2							0,928924	0,359867	0,087156
16 3							0,794981	0,600342	0,087156
16 4							0,600342	0,794981	0,087156
16 5							0,359867	0,928924	0,087156
16 6							0,091917	0,991945	0,087156
17 1							1,000000	0,000000	0,000000
17 2							0,965926	0,258819	0,000000
17 3							0,866025	0,500000	0,000000
17 4							0,707107	0,707107	0,000000
17 5							0,500000	0,866025	0,000000
17 6							0,258819	0,965926	0,000000
17 7							0,000000	1,000000	0,000000
…									

Other geometric parameters calculation

Edge length.

Each edge of derivative polyhedra was marked by consecutive number (Fig.6) – as there is a symmetry, only its one half were taken into account.

Basing on the rectangular coordinates and applying the following formula:

$$\sqrt{(A_x - B_x)^2 + (A_y - B_y)^2 + (A_z - B_z)^2} \qquad (2)$$

their lengths were calculated (Table.5).

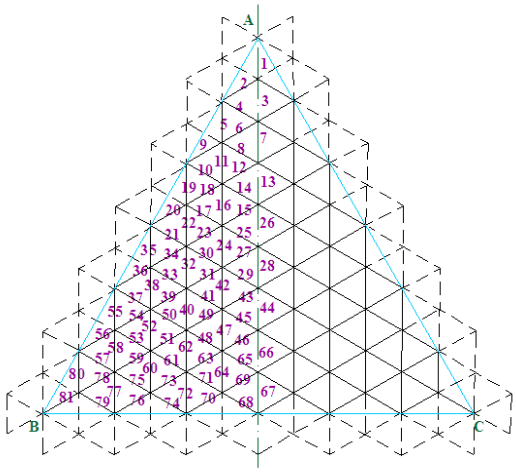

Fig.6. Marked edges (for 1/16 part) of derivative polyhedra

Table.5. Length of edges of octahedron-based polyhedra

Edge number	24-hedron	96-hedron	216-hedron	384-hedron	600-hedron	864-hedron	…
1	1,000000	0,517638	0,347296	0,261052	0,209057	0,174311	

2	0,880486	0,524648	0,361331	0,274098	0,220436	0,184222	
3		0,517638	0,347297	0,261053	0,209057	0,174312	
4		0,653351	0,467603	0,359438	0,290832	0,243855	
5		0,578626	0,397627	0,301584	0,242548	0,202710	
6		0,500498	0,371109	0,288897	0,235134	0,197786	
7		0,517638	0,347296	0,261052	0,209056	0,174312	
8		0,527928	0,418330	0,333084	0,273899	0,231675	
9		0,403361	0,308746	0,243406	0,199310	0,168211	
10			0,437684	0,358079	0,298086	0,253796	
11			0,365575	0,277291	0,223071	0,186473	
12			0,348884	0,282969	0,234730	0,199494	
13			0,347297	0,261052	0,209057	0,174312	
14			0,366397	0,306955	0,258390	0,221336	
15			0,312246	0,267649	0,227877	0,196431	
16			0,354781	0,268789	0,216176	0,180701	
17			0,367304	0,319427	0,273440	0,236333	
18			0,298412	0,246303	0,206104	0,176020	
19			0,391650	0,302789	0,245949	0,206726	
20			0,261209	0,218765	0,184555	0,158359	
21				0,323210	0,282026	0,246239	
22				0,282392	0,229406	0,192900	
23				0,236881	0,203905	0,176874	
24				0,264997	0,213023	0,178036	
25				0,279152	0,242287	0,211009	
26				0,261053	0,209057	0,174310	
27				0,246742	0,217064	0,190504	
28				0,261052	0,209057	0,174312	…
29				0,249851	0,225095	0,200087	
30				0,282904	0,250878	0,221002	
31				0,220491	0,196382	0,173649	
32				0,272220	0,220868	0,185674	
33				0,280542	0,254819	0,227338	
34				0,212514	0,184745	0,161220	
35				0,294610	0,242384	0,205368	
36				0,193115	0,169244	0,148499	
37					0,254991	0,230707	
38					0,228979	0,193996	
39					0,179269	0,159630	
40					0,216058	0,181508	
41					0,228903	0,206336	
42					0,211365	0,176609	
43					0,203549	0,182548	
44					0,209057	0,174312	
45					0,206942	0,188443	
46					0,188267	0,173091	
47					0,210419	0,175771	
48					0,207173	0,191791	
49					0,185438	0,167701	
50					0,229229	0,209812	
51					0,170169	0,155048	
52					0,221006	0,187039	
53					0,226678	0,210967	
54					0,165039	0,147867	
55					0,235795	0,201540	
56					0,153175	0,137931	
57						0,210158	
58						0,192181	
59						0,144215	
60						0,182596	
61						0,192975	
62						0,178948	
63						0,159851	
64						0,175251	
65						0,176164	
66						0,174311	

67						0,174311	
68						0,163452	
69						0,162562	
70						0,150705	
71						0,177241	
72						0,177310	
73						0,148398	
74						0,176606	
75						0,192313	
76						0,138478	
77						0,186015	
78						0,134914	
79						0,190078	
80						0,196453	
81						0,126924	
…							

On the basis of calculated lengths, the range between minimum and maximum edge length was analyzed for each derivative polyhedron. The results were tabulated in Table 6 and shown in Fig.7.

Table.6. The range between minimum and maximum edge length

polyhedra	range
24-hedron	**0,533728**
96-hedcron	**0,362006**
216-hedron	**0,256430**
384-hedron	**0,197066**
600-hedron	**0,159693**
864-hedron	**0,134128**

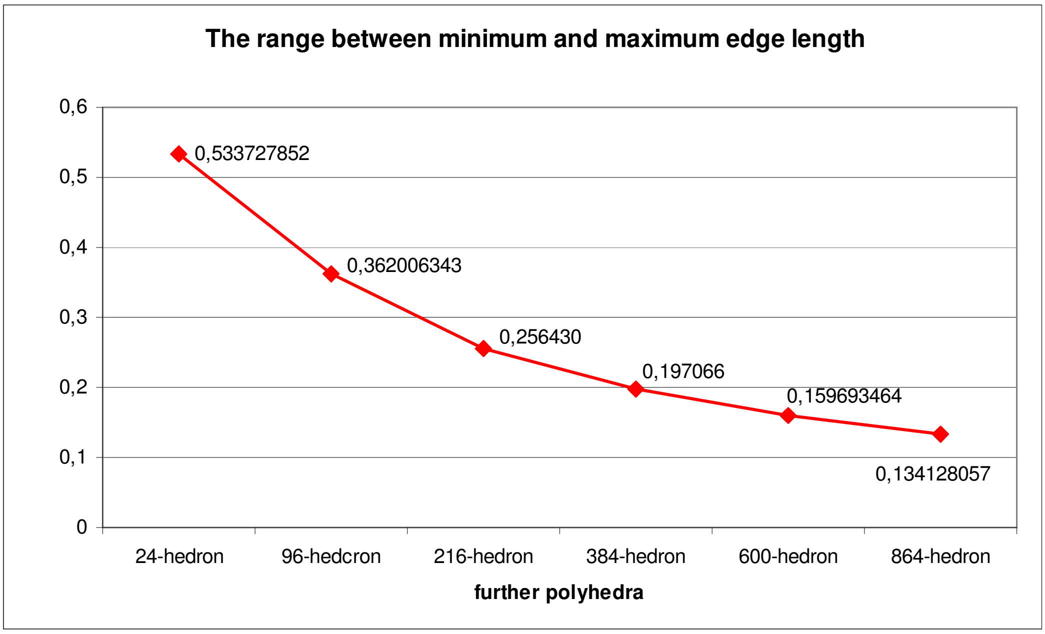

Fig.7. The range between minimum and maximum edge length

The edge lengths of consecutive regular octahedron-based polyhedra were also used as initial data for number of different edges groups calculation which were tabulated in Table 7 and shown in Fig.8.

Table.7. Number of different edges groups

	24-hedron	96-hedron	216-hedron	384-hedron	600-hedron	864-hedron
Number of different edges groups	3	10	21	37	57	82

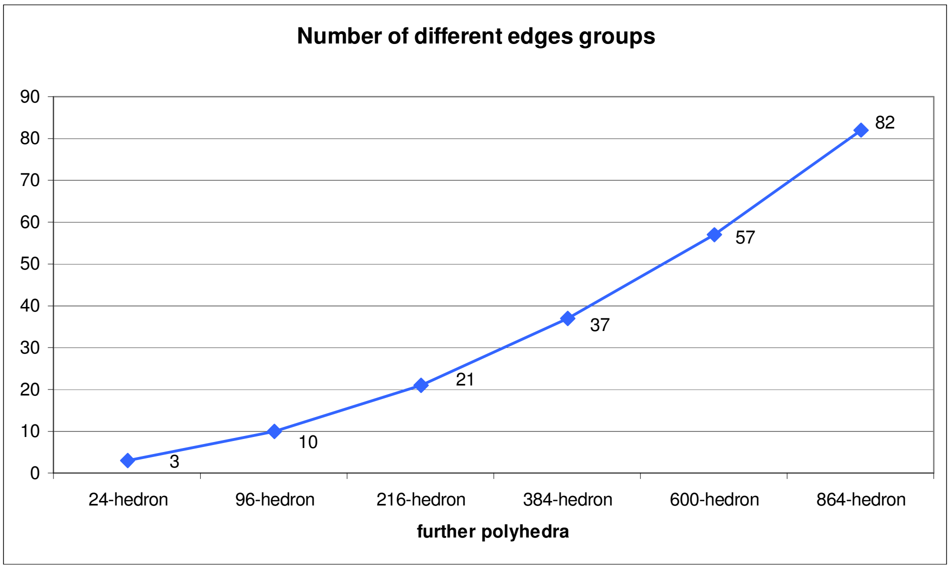

Fig.8. Number of different edges groups

Using obtained number of different edges groups, the original formula (3) for their calculation was established.

$$\mathrm{h} \longrightarrow \mathbf{G}_{\mathrm{k.r.d.}} = \frac{3}{32} \times \mathrm{h} + \frac{1}{8}\left(7 - (-1)^{\sqrt{\frac{h}{24}+1}}\right) \ldots \tag{3}$$

where: h – ...hedron, e.g. 24-hedron, 96-hedron, 216-hedron etc.

Attention: (– 1) has root in its power.

Areas.

Using rectangular coordinates and the formulas (4) and (5):

$$S^2 = \frac{1}{4}\left\{ \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2 + \begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix}^2 + \begin{vmatrix} z_1 & x_1 & 1 \\ z_2 & x_2 & 1 \\ z_3 & x_3 & 1 \end{vmatrix}^2 \right\} \tag{4}$$

$$\begin{aligned} S = \Big(\frac{1}{4}(((A_x \cdot B_y) + (A_y \cdot C_x) + (B_x \cdot C_y) - (B_y \cdot C_x) - (A_x \cdot C_y) - (A_y \cdot B_x))^2 + \\ + ((A_y \cdot B_z) + (A_z \cdot C_y) + (B_y \cdot C_z) - (B_z \cdot C_y) - (A_y \cdot C_z) - (A_z \cdot B_y))^2 + \\ + ((A_z \cdot B_x) + (A_x \cdot C_z) + (B_z \cdot C_x) - (B_x \cdot C_z) - (A_z \cdot C_x) - (A_x \cdot B_z))^2)\Big)^{1/2} \end{aligned} \tag{5}$$

areas of one face and enclosed regular octahedron-based polyhedra were calculated (Table 8). Obtained results were compared to the sphere area, on which vertices of polyhedra are lying (Fig.9).

Table.8. Area coefficient of one face and enclosed polyhedra and their sphere.

polyhedra	face	polyhedra	sphere
24-hedron	1,244267	9,954138	12,566371
96-hedron	1,473769	11,790151	
216-hedron	1,526060	12,208477	
384-hedron	1,545276	12,362208	
600-hedron	1,554350	12,434798	
864-hedron	1,559328	12,474625	

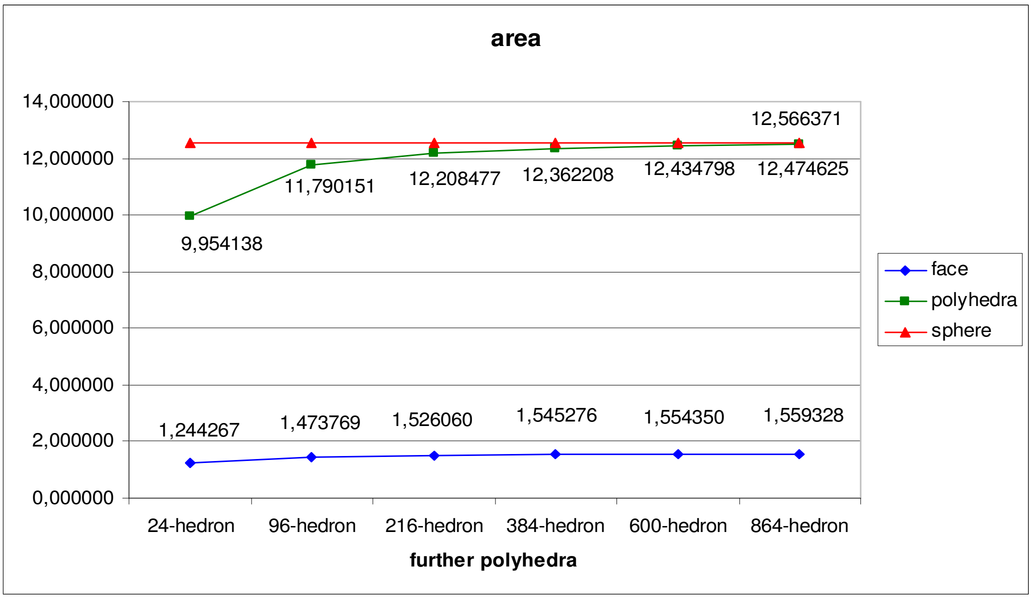

Fig.9. Area coefficient of one face and enclosed polyhedra and their sphere.

Having used calculated face areas, resulting from the division of octahedron for consecutive polyhedra, the number of different area groups were tabulated in Table 9 and shown in Fig.10.

Table.9. Number of different area groups

	24-hedron	96-Hedron	216-hedron	384-hedron	600-hedron	864-hedron
Number of different area groups	2	6	14	24	38	54

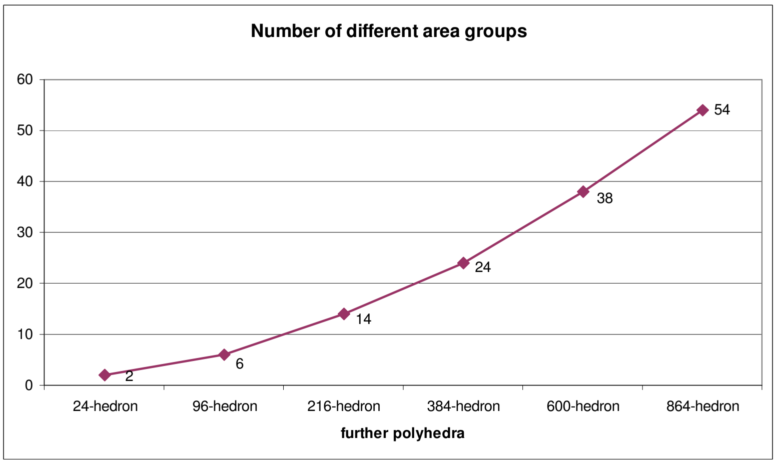

Fig.10. Number of different area groups

The above data was used to generate the original formula (6) relating to number of different area groups calculation.

$$h \rightarrow \mathbf{A}_{\text{p.p.o.r.}} = \frac{h}{16} + \frac{1}{4}\left(1 + (-1)^{\sqrt{\frac{h}{24}}+1}\right) \qquad (6)$$

where: h – ...hedron, e.g. 24-hedron, 96-hedron, 216-hedron etc.
Attention: (– 1) has root in its power.

Volume.

Assuming the engineering use of polyhedra and the fact that each edge is the bar with the same section area I = 1, the volume of one triangular face ABC and the whole polyhedra was calculated. The results were illustrated in Fig.11.

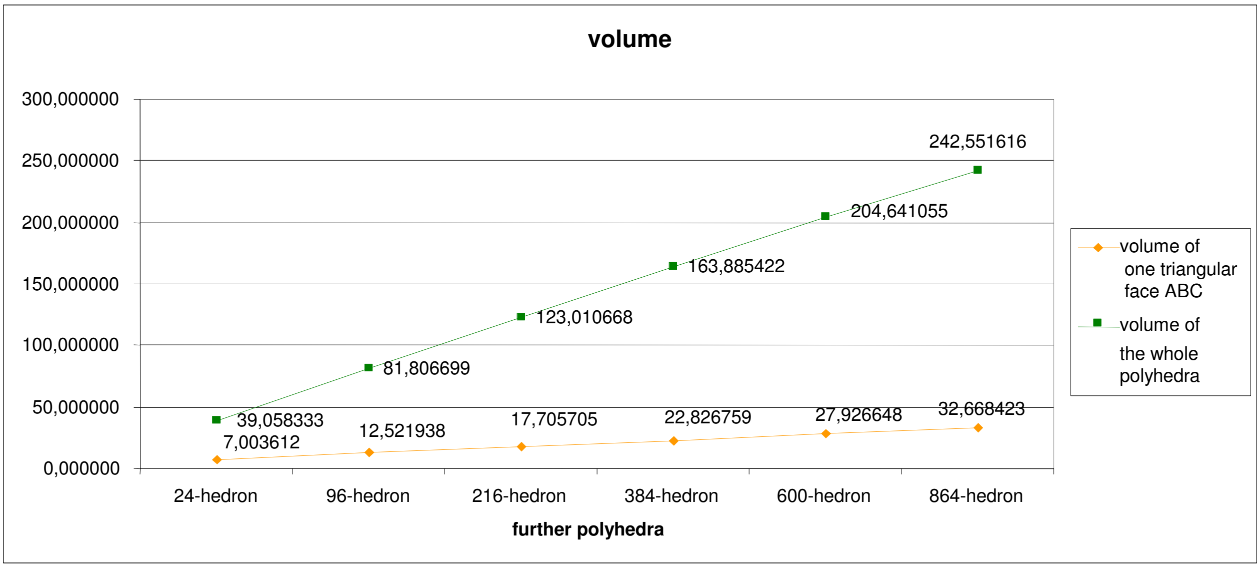

Fig.11. The edge volume of one face ABC and the whole regular octahedron-based polyhedra.

Conclusions

On the basis of the second method of subdivision of each equilateral face into smaller triangular elements the family of regular octahedron-based polyhedra was generated. The calculated angular and rectangular coordinates, number of different edge and area groups, areas of one face ABC and the whole polyhedra, range between minimum and maximum edge length and volume – allowed creation of original formulas and curves illustration.

Geometric parameters for 6 derivative polyhedra presented in this study are enough for preparing final conclusions. Generated polyhedra can be treated as the topology [5] of frame structures when it is assumed that the resulting vertices and edges constitute the nodes and members, respectively.

The calculation of geometric parameters focused on in this study, for other polyhedra is possible by applying it to the given methodology. Presented method of geometric parameters of regular octahedron-based polyhedra can determine the base of choice of the proper polyhedra for designers.

References:

[1]. Wygodski M.J. i inni: Zbiór artykułów: *O elementach Euklidesa*. Tłumaczyła z rosyjskiego J. Turkowska. PWN, Warszawa 1956.
[2]. Fuliński J.: *Geometria kratownic powierzchniowych*. Prace Wrocławskiego Towarzystwa Naukowego. Seria B; nr 178/1973.
[3]. Mirski J.Z.: *Siatki powstałe z przekształceń 8-ścianu foremnego*. Zeszyty Naukowe Akademii Rolniczej we Wrocławiu. Melioracja XVI; nr 212/1992, 27-39.
[4]. Bysiec D., Mirski J.Z.: *Determining the geometric parameters of frame octahedron-based geodesic domes*. XIV LSCE. Local Seminar of IASS Polish Chapter. Warsaw 5 December, 2008, 28-33. Micro-Publisher J.B. Obrębski W.N.
[5]. Majid K.I.: *Optymalne projektowanie konstrukcji*. Tłumaczyli z angielskiego Brzeziński J. & R. Pyrz. PWN, Warszawa 1981.

SIATKI SFERYCZNE POWSTAŁE NA BAZIE DRUGIEJ RODZINY PODZIAŁÓW OŚMIOŚCIANU FOREMNEGO

Celem pracy jest wyznaczenie parametrów geometrycznych wielościanów pochodnych od oktaedru, wygenerowanych na podstawie II sposobu podziału trójkąta równobocznego. Skoncentrowano się na 6 wielościanach pochodnych i na ich podstawie ustalono sposób obliczania analizowanych parametrów geometrycznych.

ISSN 1644-9363 / PLN 15.00