THE EQUILATERAL TRIANGLES OF A GIVEN SIDE WHOSE VERTICES BELONG TO THREE NON COMPLANAR STRIGHT LINES

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Abstract: This paper is continuation of the problem originated in the paper [2], and contains the analytic proof of the statement, which determines the set of points of plane, which are the vertices of equilateral triangles of given side and two residual (remaining) belonging to two intersecting lines. This statement is used to the prove that in the 3-dimensional space set of these points there is kinetic surface of stable generating line which is in the shape of circle. In the case of two parallel lines this surface is elliptic cylinder. The construction of the common points of the third line and in this manner obtained kinetic surface make possible the discovery of the vertices of equilateral triangle of given side which the vertices belong to the three given non coplanar straight lines.

Keywords: equilateral triangle, kinetic surface, elliptic cylinder surface

1 Introduction

In the paper [2] the construction of equilateral triangles are given, which the vertices belong, respectively, to three straight lines belonging to the same plane. Moreover for three lines belonging to the same plane (forming closed/open triangle) utilizing non proved in the work [2] the theorem of the effective construction of equilateral triangle about given side and inscribed in this triangle has been presented. This construction can to be applied also in the case of three coplanar lines which belong to the same point. The aim of this work is to present the proof of theorem which was prospective in the paper [2] and which will be used also to construct the equilateral triangles of given side and the vertices belonging, respectively, to given three non coplanar lines.

2 The main Theorem

Theorem If the endpoints of the segment $d \neq 0$ which is common side of two equilateral triangles, translate on the two of intersecting lines, then their third vertices belonging to the same plane, describe two of concentric ellipses with the center in the point of intersection of these lines and length of axes: $2a_1 = d(c/tg\frac{\alpha}{2}+\sqrt{3})$ and $2b_1 = d(\sqrt{3}-tg\frac{\alpha}{2})$ and also $2a_2 = d(tg\frac{\alpha}{2}+\sqrt{3})$ and $2b_2 = d(ctg\frac{\alpha}{2}-\sqrt{3})$ which belong to the angles bisectors of these lines (α and $180^o - \alpha$).

These ellipses are the sets of vertices of equilateral triangles which two remaining vertices belonging , respectively, to the intersecting lines.

Let us assume two lines a and b on a plane belonging to the same point O forming the angles α and $180^o - \alpha$, and also one of the locations of the segment $A_iB_i = d$ whose endpoints

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belong to these lines, respectively (Fig.1). Assuming lines a and b as axes of oblique system of reference Oxy with the angle α, coordinates of the points $A_i \in a = x$ are $(x_1,0)$ and the points $B_i \in b = y$ - $(0,y_2)$. The coordinates of the midpoints S_i of the segment A_iB_i are equal:

$$x_3 = \frac{1}{2}x_1 \text{ and } y_3 = \frac{1}{2}y_2, \text{ hence} \tag{1}$$

$$x_1 = 2x_3\ ;\ y_1 = 0 \text{ and } x_2 = 0\ ;\ y_2 = 2y_3 \tag{2}$$

Now we show that

Lemma 1 The midpoints S_i of the segments $A_iB_i = d$ whose endpoints translate, respectively, on the lines a and b belonging to the same point O and forming an angle $\alpha \neq 0$, describe the ellipse with the center in the point O= a·b and the axes of length: $2a = d\,\mathrm{ctg}\frac{\alpha}{2}$ and $2b = d\,\mathrm{tg}\frac{\alpha}{2}$ which belong to the bisectors of the angles of the lines a and b.

Proof: In the oblique of coordinate system Oxy the formula of length of the segment is the following:

$$A_iB_i = d = \pm\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + 2(x_2 - x_1)(y_2 - y_1)\cos\alpha}\,. \tag{3}$$

Introducing the relations (2) into the equation (3), after arrangement and abandoning the indexes we obtain the equation:

$$\left(\frac{d}{2}\right)^2 = x^2 + y^2 - 2xy\cos\alpha. \tag{4}$$

Next on that same plane let us assume a new coordinate system (oblique or orthogonal) whose point O= O' and $\angle x,x' = \varphi$, and $\angle y,y' = \psi$. If $\psi = \varphi + \pi$ then the axes x' and y' are perpendicular lines [1]. The change of the oblique coordinate system Oxy with the angle α to another coordinate system (oblique or orthogonal) with this same beginning O = O' is realized by the relations occurring between coordinates x,y and x',y' of any point successive in relation to the systems Oxy and Ox'y':

$$x = \frac{x'\sin(\alpha - \varphi)}{\sin\alpha} + \frac{y'\sin(\alpha - \psi)}{\sin\alpha} \text{ and } y = \frac{x'\sin\varphi}{\sin\alpha} + \frac{y'\sin\psi}{\sin\alpha}. \tag{5}$$

In this case for $\varphi = \frac{\alpha}{2}$ and $\psi = \frac{\alpha + \pi}{2}$ the relations (5) have the following form:

$$x = \frac{x'\sin\frac{\alpha}{2}}{\sin\alpha} - \frac{y'\cos\frac{\alpha}{2}}{\sin\alpha} \text{ and } y = \frac{x'\sin\frac{\alpha}{2}}{\sin\alpha} + \frac{y'\cos\frac{\alpha}{2}}{\sin\alpha}. \tag{6}$$

Introducing the relations (6) instead of x and y into the equation (4) after the proper trigonometric transformation we obtain the equation of the ellipse:

$$\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1, \tag{7}$$

where:

$$a = \frac{d}{2}\mathrm{ctg}\frac{\alpha}{2} \text{ and } b = \frac{d}{2}\mathrm{tg}\frac{\alpha}{2}. \tag{8}$$

Thus, in the orthogonal coordinate system Ox'y' whose axes are bisectors of the angles of different lines a and b belonging to this same point O, the midpoints S_i of the segments

A_iB_i = d, describe the ellipse e_s (Fig.1). When $\alpha = \frac{\pi}{2}$ then we obtain the circle with center O and the radius $r = \frac{d}{2}$ ($tg45^o = ctg45^o = 1$ and $a = b = r = \frac{d}{2}$).

Carrying on the proof of the fundamental theorem, we consider the perpendicular bisector n_i of the privileged segment A_iB_i and assume any point $E_i \neq S_i$ ($S_iE_i = h \neq 0$). We shown that

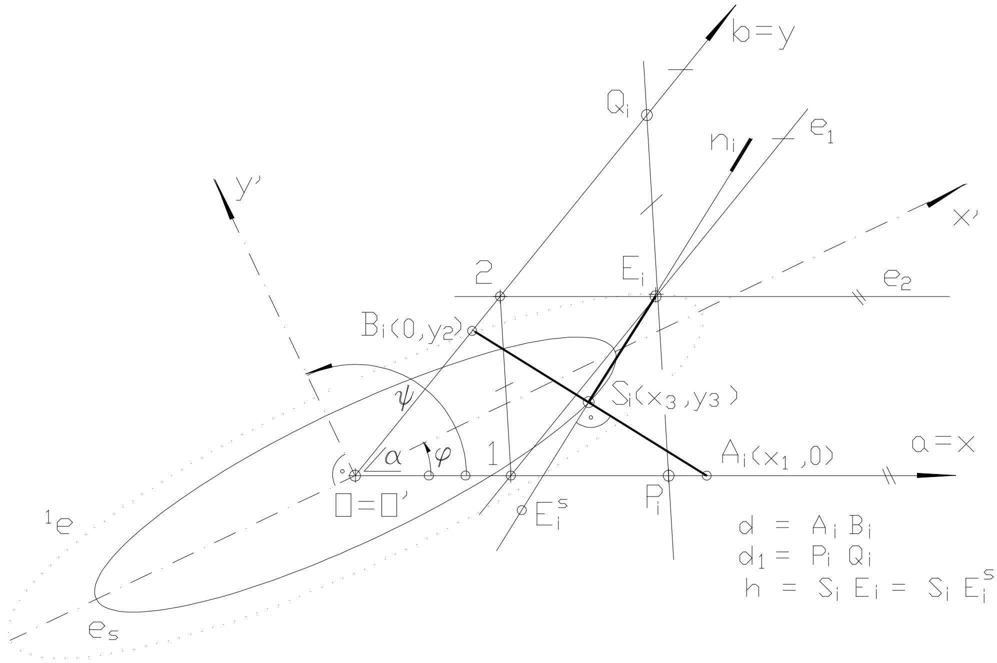

Fig. 1

Lemma 2 *Every point of a plane determined by an intersecting lines, except the points belonging to these lines, is the midpoint of the segment whose endpoints belong to these lines.*
Proof: The proof carried by construction of this statement is well known. The lines $e_{1,2}$ passing through the point E_i and parallel to the lines b and a respectively, intersect them in the point 2 and 1 (Fig.1). Next the line belonging to the point E_i and parallel to the line 12 intersects the lines a and b, respectively at the points P_i and Q_i which are the endpoints of the segment $P_iQ_i = d_1$. The segment 12 and point E_i are the common elements of two parallelograms: $12P_iE_i$ and $12Q_iE_i$. The conclusion is that $12 = P_iE_i$ and $12 = Q_iE_i$, hence $P_iE_i = Q_iE_i$ and ipso facto the point E_i is the midpoint of the segment P_iQ_i. According to lemma 1 point E_i generates concentric and coaxial ellipse ^1e in relation to the ellipse e_s whose length of the axes are : $2a_1 = d_1 ctg\frac{\alpha}{2}$ and $2b_1 = d_1 tg\frac{\alpha}{2}$. Let us point out that $a_1 = a + h$, $b_1 = h - b$ and the segment $S_iE_i = h = \frac{d_1 - d}{2} ctg\frac{\alpha}{2}$ for two different points of line n_i is the stable value. Analogously, point E_i^s of the lines n_i symmetrical to point E_i in relation to point/segment S_i/A_iB_i as the midpoint of the segment d_2, describes the second ellipse ^2e whose axes are equal : $2a_2 = d_2 tg\frac{\alpha}{2}$ and $2b_2 = d_2 ctg\frac{\alpha}{2}$ and belong to the bisectors of angles as form the lines a and b ($a_2 = b + h$ and $b_2 = a - h$). Fig.1 neglects this ellipse due to its clearance. These ellipse are the sets of vertices of isosceles

triangles with the height h and of the common foundation (base) A_iB_i. For $h = \frac{d\sqrt{3}}{2}$ we obtain on the lines n_i the points ${}^{1,2}C_i$ which are the vertices of equilateral triangles of the common side A_iB_i. Due to lemmas 1 and 2 these points as the midpoints of the segments whose end-points belong to lines a and b, describe two of concentric and coaxial ellipses ${}^{1,2}e$ in relation to the ellipse e_s of length axes: $MN= 2a_1= d(\mathrm{ctg}\frac{\alpha}{2}+\sqrt{3})$ and $OP= 2b_1=d(\sqrt{3}-\mathrm{tg}\frac{\alpha}{2})$ plus $VW=2a_2=d(\mathrm{tg}\frac{\alpha}{2}+\sqrt{3})$ and $TU= 2b_2= d(\mathrm{ctg}\frac{\alpha}{2}-\sqrt{3})$ belonging to the bisectors of the angles of the lines a and b – as it was necessary to prove.

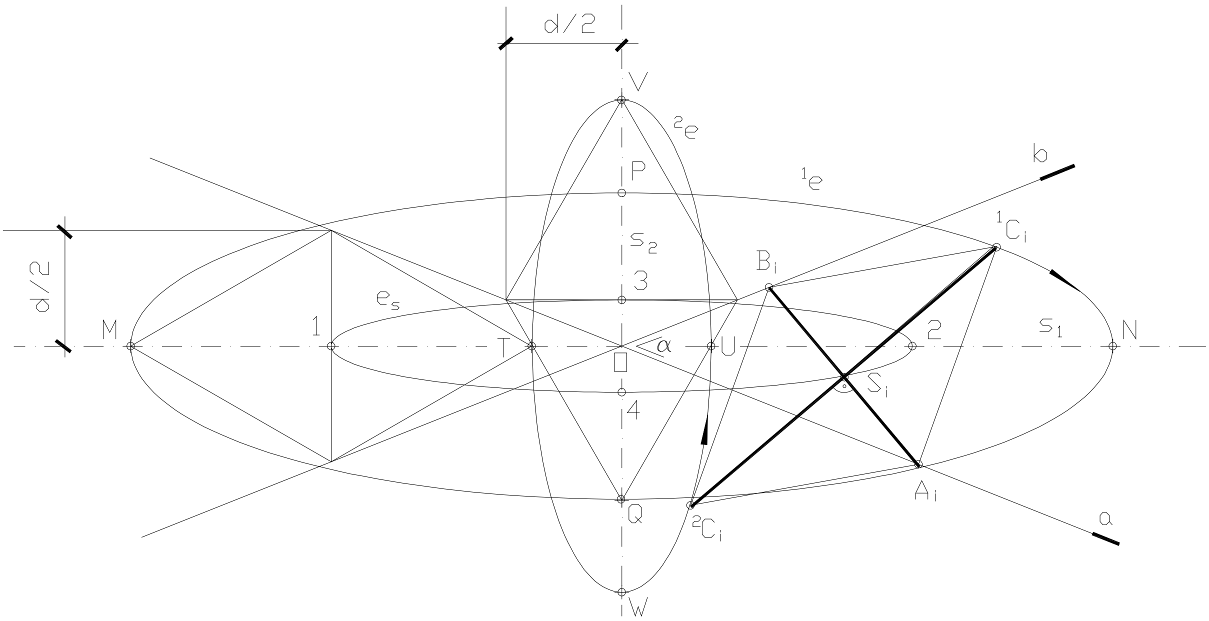

Fig. 2

Fig. 2 shows construction axes of these two ellipses for given segment d – the side of the equilateral triangle. While the angel $\boldsymbol{\alpha} = 60^{\circ}$ that ellipse ${}^{2}e$ degenerates to the segment $VW= d(\mathrm{tg}\frac{\alpha}{2}+\sqrt{3})$, because $\mathrm{ctg}30^{\circ}=\sqrt{3}$, thus the segment TU=0. For the angle $\boldsymbol{\alpha}=\frac{\pi}{2}$ the ellipses ${}^{1,2}e$ are identical figure and symmetrical in relation to lines a and b.

3 The equilateral triangles of a given side whose vertices belong to three of non coplanar lines

In the second part of this paper the author gives the construction of the equilateral triangle of a given side whose the vertices belong to non coplanar of three lines. For two (out of three) of privileged lines determined in the 3-dimensional space the set of points being the vertices of the equilateral triangles of given side whose two other (remaining) lie on these privileged lines. The common points of this set and third line form the vertices which were looked for. Out of possible locations of three given skew lines in the space, it is necessary to consider only three events:

- two intersecting lines and the third non coplanar with them,
- two parallel lines and third non coplanar with them,

- three of the pairs skew lines.

3.1 Two intersecting lines and the third one non coplanar with them

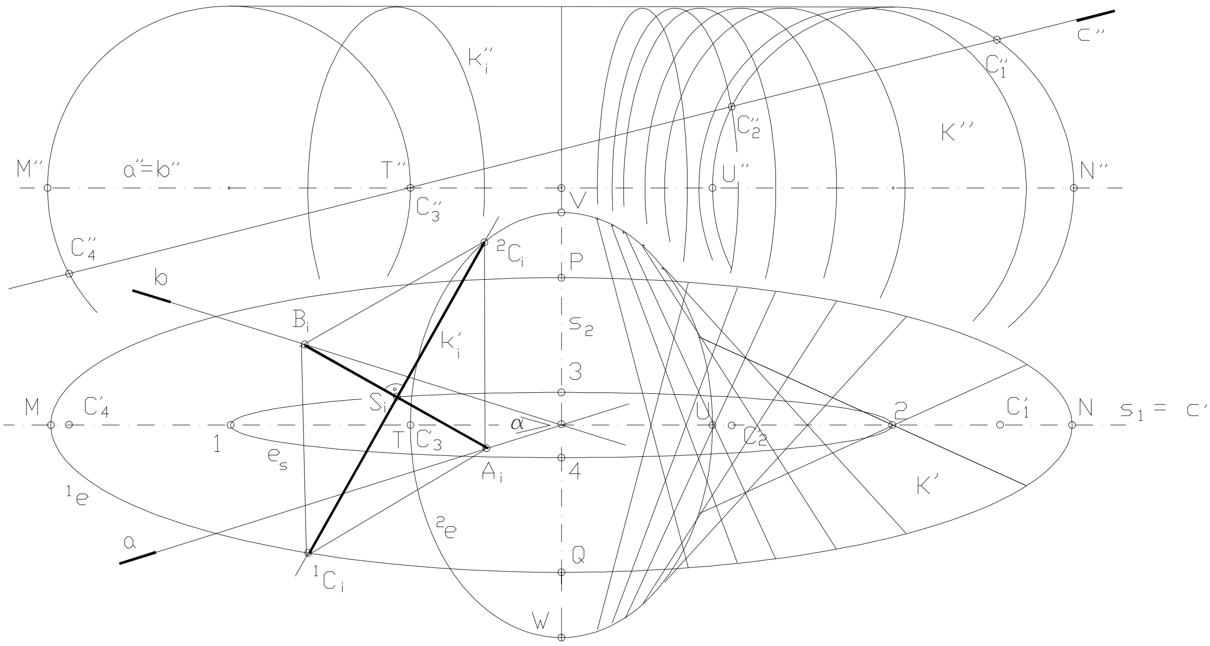

Fig. 3

Let us assume that out of three lines a, b and c, the lines a and b intersecting in the point O lie on the plane of draw τ identify with horizontal (first) plane of projection π_1, whereas third line c does not lie on this plane. For the given length of the side equilateral triangle d≠0 we designate the axes of the ellipses $^{1,2}e$ (as shown in Fig. 2) being on the plane (a and b) the sets of the vertices of equilateral triangles of side d whose two other vertices belong to these lines, respectively (Fig. 3). Moreover, the axes of ellipse e_s were denoted. From of one locations of segment A_iB_i= d whose endpoints belong to the lines a and b, correspond on the plane of the drawing to two points $^{1,2}C_i$ belonging, respectively, to the ellipses $^{1,2}e$, and in the 3-dimensional space, the set of these points which together with of the points A_i and B_i form the equilateral triangles is the circle k_i with center S_i and radius $r = \frac{d}{2}\sqrt{3}$ which lies on the plane perpendicular to the bisector of the segment A_iB_i. Conjugate diameter of this circle to its the diameter $^1Ci\ ^2C_i$ belongs to the generating line of elliptic cylinder which right section is the ellipse e_s (Fig. 3). While the segment A_iB_i= d moves in this way that its endpoints translate, respectively, on the lines a and b then the circles $k_i(S_i,\ r=\frac{d}{2}\sqrt{3})$ included on the planes perpendicular to the segments A_iB_i, generate the kinetic surface K. The ellipse $^{1,2}e$ are the directrix of this kinetic surface. According to classification and terminology of kinetic surfaces presented in [3] and [4] the obtained kinetic surface K can be called rotation / rotary. The points of intersection of the lines c and in this way obtained kinetic surface K are the looked for vertices of equilateral triangles of given side d whose remaining two vertices lie on the lines a and b, respectively. While the line c passes through the point O = a·b that we obtain the tripod Oabc and its sections in the shape of equilateral triangles with given side. Fig. 3 shows the construction of common points of the line c and the surface K. And Fig.4 is a understandable illustration of kinetic surface formed by two intersecting lines a and b which include the angle equal 60^o (in this case one of directrix of surface K degenerates the segment).

3.2 Two parallel lines and third one which is non coplanar

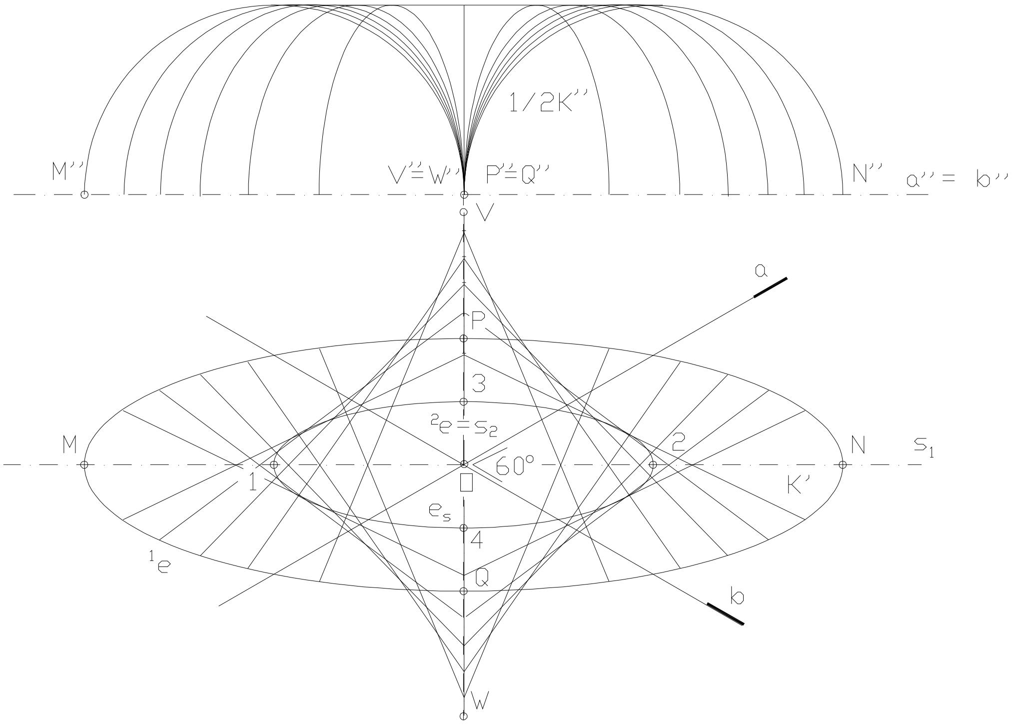

Fig. 4

Now let us assume that out of three lines a, b and c the lines a and b are parallel and lie on the plane of the drawing $\tau=\pi$, and the distance between them is equal x. Let us consider one of positions of the segment $A_iB_i = d \geq x$ whose endpoints lie ,respectively, on the lines a and b , and also the points $^{1,2}C_i$ being vertices of equilateral triangles $^{1,2}C_iA_iB_i$ with the common side (Fig. 5). If the segment A_iB_i moves in the way that its endpoints translate, respectively, on the lines a and b then its midpoint S_i describes the line e_s equidistant from these lines, and the points $^{1,2}C_i$ – two lines $^{1,2}e\|e_s$ and symmetrical relative to its (the properties of translation).

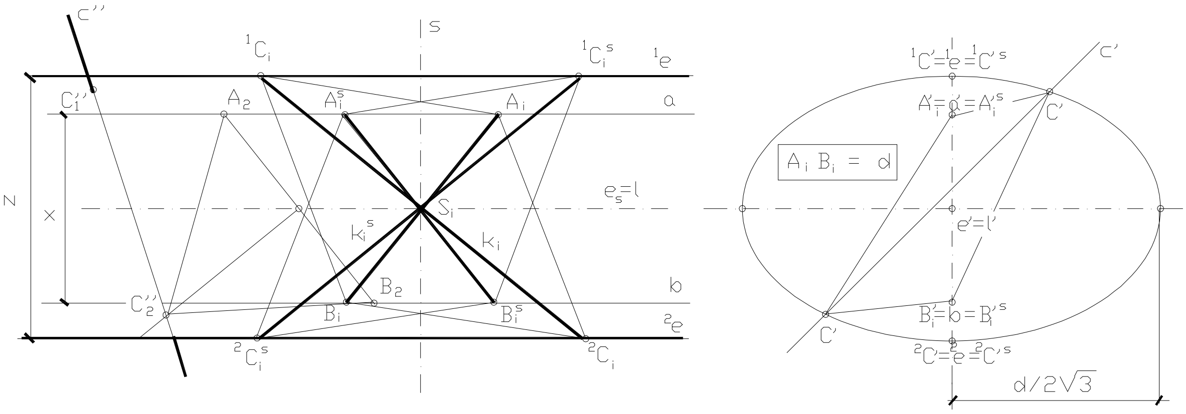

Fig.5

For the segment A_iB_i= d two such points $^{1,2}C_i$ exist on the plane of the drawing, which together with the points A_i and B_i form the equilateral triangles with common side. In the 3-dimensional space a set of such points is the circle k_i with center S_i and radius $r=\frac{d}{2}\sqrt{3}$ included in the plane perpendicular to the segment A_iB_i (Fig. 5). Diameter of this circle conjugates to its diameter ^1C_i ^2C_i and is perpendicular to the plane of the drawing. The circles k_i and the circles k_i^s symmetrical to them in relation to the line s⊃ ($S_i \in$ s) and perpendicular to the line e_s, form the kinetic surface called translation, which in this case is surface of elliptic cylinder. The common points of this surface (elliptic cylinder) and third line c are the vertices of equilateral triangles which comply with conditions of assumptions (Fig. 5). If line c is one from generating lines of elliptic cylinder then we obtain the tripod O^∞abc and its sections in the shape of equilateral triangles. As the case from relations between x and d the locations lines $^{1,2}e$ (distance between them equal z) relative to the lines a and b can be following. While $d=\frac{2}{3}x\sqrt{3}$, that z = x (lines ^1e = a and ^2e = b). While d = x that z=0 and then the lines $^{1,2}e$ = e_s(equidistant line from the lines a and b). In the case d$<$ x that z $<$ x and while d $>$ x than also z $>$ x (Fig. 6).

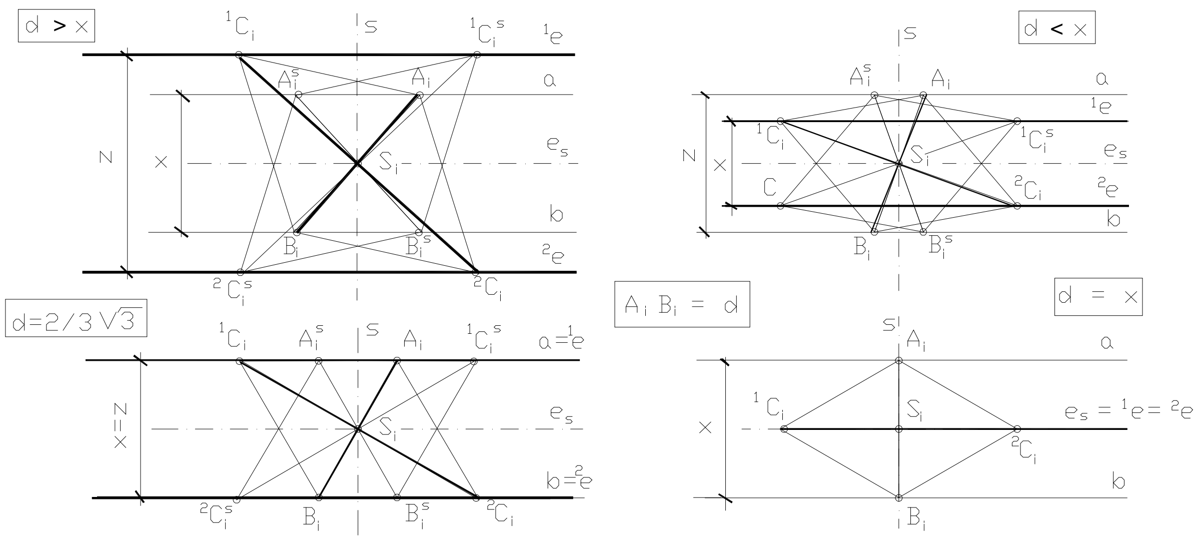

Fig. 6

From the above considerations we can obtain the simple construction of the equilateral triangle of a given side inscribed in the open triangle. For that purpose we construct the auxiliary equilateral triangle ABC of given side equal d, which the vertices A and B lying respectively, on the lines a and b. The line ^1e including the point C and parallel to the line a‖b intersects the line c at the point C_1. Next the lines $t_{1,2}$ intersecting point C_1 and parallel, respectively, to the sides CA and CB of the auxiliary triangle ABC, intersect the lines a and b in the points A_1 and B_1. The triangle $A_1B_1C_1$ is the identical figure to the auxiliary triangle ABC. The second solution is point C_2 in which line ^2e is symmetrical to the line ^1e relative to the line e_s intersects line c.

3.3 Three of the pairs of skew lines

Out of three given skew lines a, b and c we select the pair of lines a i b belonging to their distance planes σ and μ respectively. Moreover we assume that the plane σ unites with the first projection plane $\boldsymbol{\pi}_1$. Each two skew lines of 3-dimensional space with the proper change of the systems of reference (Fig. 7) can be reduced to this location.

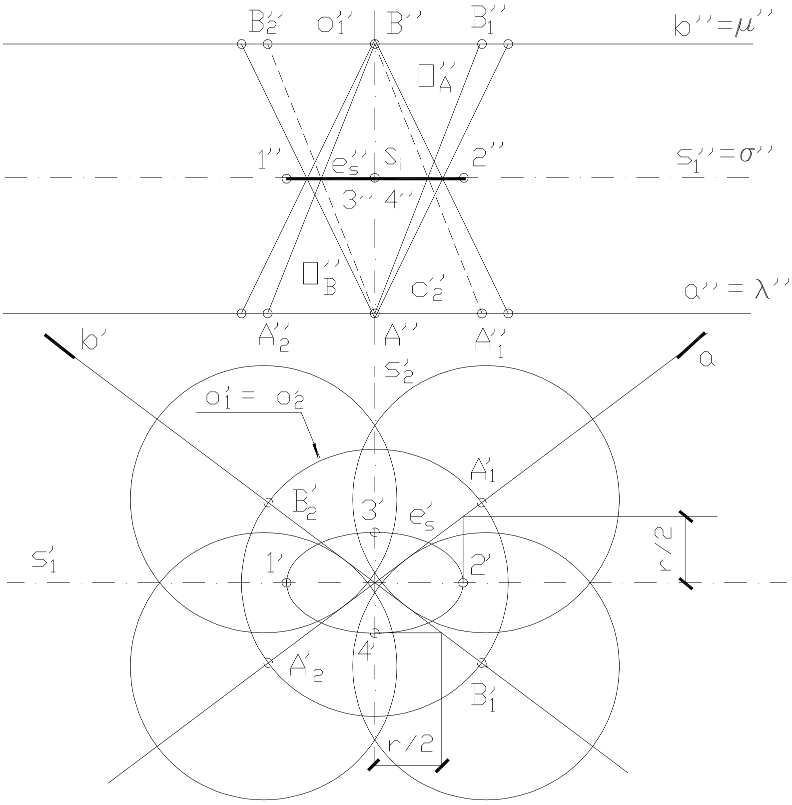

Fig. 7

It is necessary to point out that the 3-dimensional space has two axial symmetries which transform line a to line b and vice versa. The axes of these symmetries are the lines $s_i(i=1,2)$ passing through the midpoint of the distance segment, perpendicular to it and positioned in the way that their of orthogonal projections on the distance planes (and the planes parallel to them) are angles bisectors forming by the orthogonal projections of lines a and b, hence are perpendicular. The distance segment of these skew lines AB=h whose endpoints lie, respectively, on them (i.e. on the lines a and b) is assumed as the common height (h) of two identical circular cones O_A and O_B with the vertices A and B and plus the length of their generating lines equal the given segment d>h. From the relation $d^2 = h^2 + r^2$ for given d and h we obtain the radius r of the circles $o_i(i=1,2)$ being the bases of these cones included on the distance planes λ and μ, respectively. From the above considerations it follows that the elements of these cones (the segments d) contained with the distance planes and ipso facto with the plane of projection π_1 are at the same angle ε. The circle o_2 being the base of cone with the vertex B intersects the line a belonging to the plane of this circle in the points $A_i(i=1,2)$. The midpoints $S_i(i=1,2)$ of the segments BA_i= d are the foots of heights of equilateral triangles with common sides whose third vertices belong to the circles ${}^bk_i(i=1,2)$ included on the planes perpendicular to the segments $BA_i(i=1,2)$. The centers of these circles are the points S_i and their radius R= $\frac{d}{2}\sqrt{3}$. By analogy the circle o_1 as directrix of the cone of vertex A intersects the line b belonging to its plane in the points $B_i(i=1,2)$. The midpoints S_i of the segments

$AB_i(i=1,2)$ are the centers of the circles ${}^a k_i(i=1,2)$ and of the radius $R=\frac{d}{2}\sqrt{3}$ belonging to the planes perpendicular to these segments (Fig. 7). Translation of cones O_A and O_B in the directions, respectively, to the lines b and a, from the early to the limits positions (contact their bases to the lines a and b) and doing the presented constructions for initial positions we obtain successive segments of the lengths d and the endpoints belonging respectively to the lines a and b. These segments correspond with the identical circles ${}^b k_i$ and ${}^a k_i$ with centers S_i and of radius $R=\frac{d}{2}\sqrt{3}$ included in the planes perpendicular to them. Midpoints of these segments being the elements of the identical circular cones of the parallel axes are included in the plane σ equidistant from the distance planes λ and $\mu\boldsymbol{\beta}$.

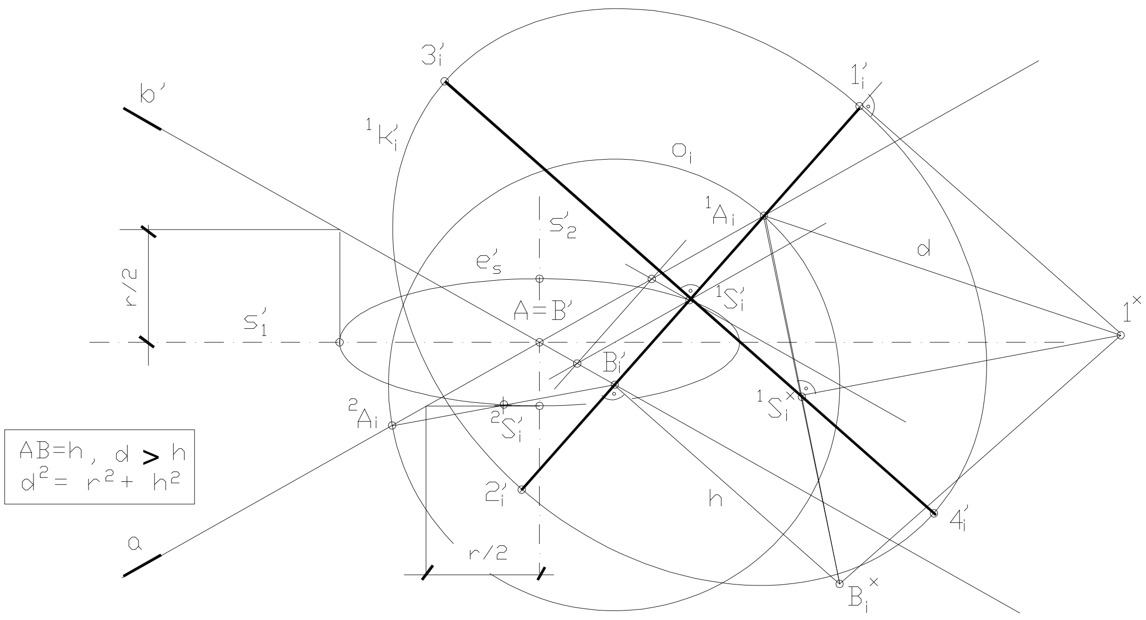

Fig. 8

Now let us consider one of the locations of segment d together with its midpoint S_i whose endpoints belong to lines a and b respectively. The first projection of this segment is segment $d' = d \cdot \cos\varepsilon\varepsilon = r$ whose endpoints belong to the projections a'= a and b' of the lines a and b, and its midpoint is projection S_i'of the point S_i. Due to the lemma 1 (point 1) if the segment d' moves in the way that its endpoints translate, respectively, on the lines a' and b' then its midpoint S_i' describes the ellipse e' with center in the point $O' = a \cdot b'$ and of the length axes : $2a = d'\mathrm{ctg}\frac{\alpha}{2}$ and $2b = d'\mathrm{tg}\frac{\alpha}{2}$ (where $d' = r$, and the angle $\boldsymbol{\alpha} = \angle(\sphericalangle ab')$). The axes of this ellipse belong , respectively, to the first projections of the axes symmetry $s_i(i=1,2)$ of the skew lines a and b. While the lines a and b are perpendicular then the ellipse e' passes the circle $k'(O', \frac{r}{2})$. From properties of projections theory it is known that degree of coplanar curve includes on the plane non parallel to direction of projection is the same as the degree of its projection and moreover its projection is identical figure to the original, while the curve lies on the plane parallel to the plane of projection. Hence the conclusion is that ellipse e' is orthogonal projection of identical to its ellipse e which belongs to the plane σ being the set of the midpoints S_i of the segments d whose endpoints belong to the skew lines a and b.

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Thus in this way it has been shown that if the endpoints of the segment d≠**0** translate, respectively, on two skew lines, then its midpoint describes the ellipse includes on the plane equidistant from the distance planes of these lines. The point of intersection of the axes of symmetry of skew lines is its midpoint, and the lengths of its axes are equal: $2a = d'\text{ctg}\frac{\alpha}{2}$ and $2b = d'\text{tg}\frac{\alpha}{2}$. These two families of identical circles ${}^{a}k_i$ and ${}^{b}k_i$ generate kinetic rotation of surface K. Each point of this surface is the vertex of equilateral triangle which comply with the conditions of the assumptions. The common points of third line c and in this way formed surface K are solution to this problem.

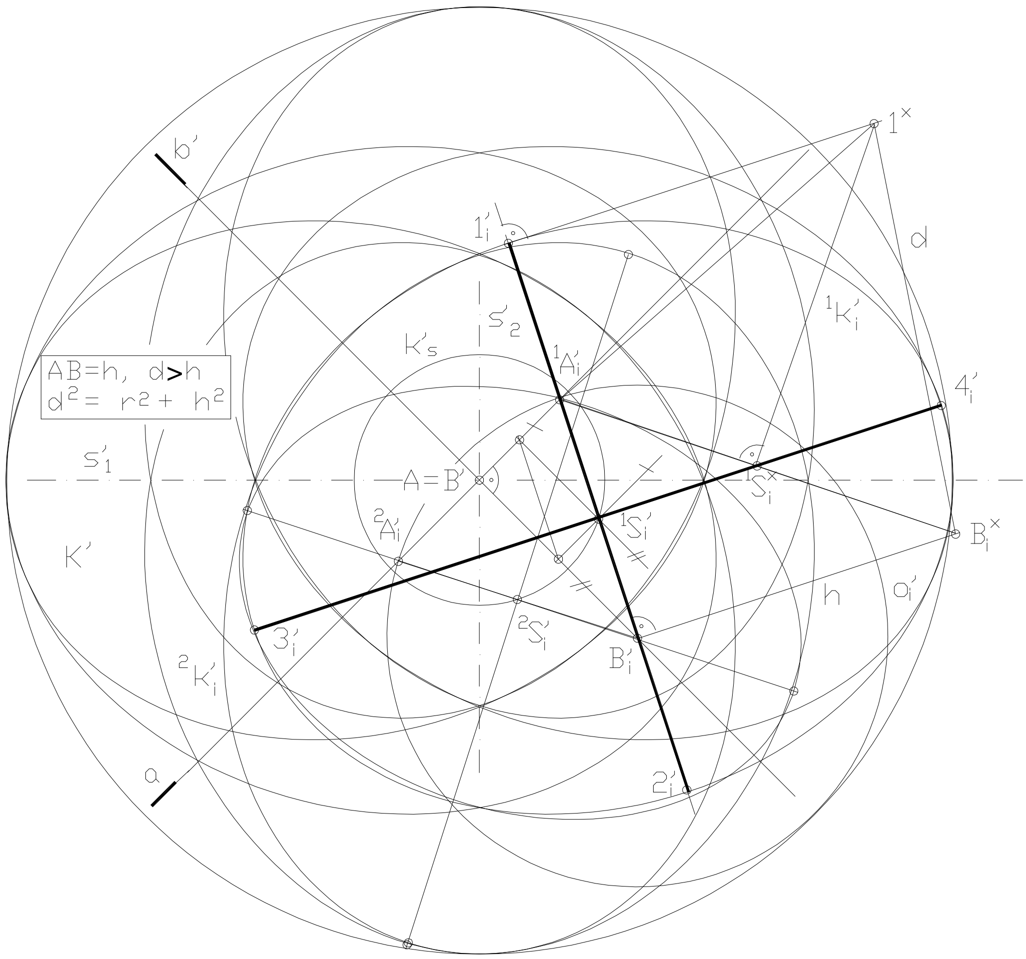

Fig. 9

The orthogonal projections of the circles (as generating lines of this surface) of this kinetic surface K are identical ellipses whose axes are equal: major axis $2a = d\sqrt{3}$ and small axis $2b = d\sqrt{3}\sin\varepsilon$ ($\varphi = \angle \alpha \supset k_i, \pi$; $\cos\varphi = \cos(90^{o} - \varepsilon) = \sin\varepsilon$) (Fig.8). Fig. 9 is orthogonal projection on the plane of kinetic surface K which is formed for the pair of perpendicular skew lines.

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TRÓJKĄTRY RÓWNOBOCZNE O ZADANYM BOKU I WIERZCHOŁKACH NALEŻĄCYCH DO TRZECH NIEKOMPLANARNYCH PROSTYCH

W pracy, która jest kontynuacją problematyki zapoczątkowanej w artykule [2] podano dowód twierdzenia określającego zbiór punktów płaszczyzny, które są wierzchołkami trójkątów równobocznych o zadanym boku i dwóch pozostałych wierzchołkach leżących odpowiednio na dwu przecinających się prostych. Twierdzenie to wykorzystano do wykazania, że w 3-wymiarowej przestrzeni zbiorem takich punktów jest powierzchnia kinetyczna o stałym kształcie tworzącej, w postaci okręgu. W przypadku dwu prostych równoległych powierzchnia ta jest powierzchnią walcową eliptyczną. Wyznaczenie punktów wspólnych trzeciej prostej z tak utworzoną powierzchnią kinetyczną prowadzi do znalezienia wierzchołków trójkąta równobocznego o zadanym boku spełniającego warunki zadania.