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A GROUP OF SPHERICAL TESSELLATIONS BASED ON REGULAR OCTAHEDRON

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Abstract. The aim of the study was to calculate of geometric parameters of polyhedra with the bigger amount of faces – derived from a regular octahedron. The algorithms were developed using the MsExcel program. The analysis was concentrated on three regular octahedron-based polyhedra with an even number of edges and on three regular octahedron-based polyhedra with an odd number of edges.

Keywords: regular polyhedra, division of equilateral triangle, regular octahedron-based polyhedra, geometric coordinates, angular coordinates, length of edges, area.

1. Introductory information

To make an analysis of geometric parameters of one of five regular polyhedra [2], octahedron, one of three possible positions of axes space coordinates was chosen and shown in Fig. 1.

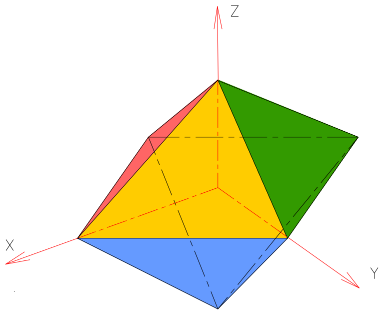

Fig. 1: Assumed position of X,Y,Z axes

Octahedron-based polyhedra can be obtained by each face division (i.e. equilateral triangle) of original octahedron by smaller ones according to one of three known methods [3, 5]. This study focuses on the first method of division.

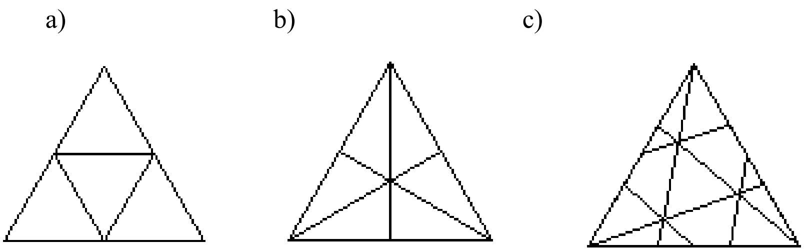

Fig. 2: Three methods of division of starting face of regular octahedron

Regular octahedron-based polyhedra were tabulated in Table 1. Polyhedra generated by applying the first method of division are presented diagonally (Fig 2a) and are analyzed in this work.

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Table 1: Regular octahedron-based polyhedra; t,k – consecutive natural numbers

t \ k	1	2	3	4	5	6	7	8	…
1	8	**24**	56	104	168	248	344	456	
2	24	32	56	**96**	152	224	312	416	
3	56	56	72	104	152	**216**	296	392	
4	104	96	104	128	168	224	296	**384**	
5	168	152	152	168	200	248	312	392	
6	248	224	216	224	248	288	344	416	
7	344					344	392	486	
8								512	
…									…

2. Geometric parameters of regular octahedron-based polyhedra

There are methods of calculation of geometric parameters: angular coordinates and rectangular coordinates, and on the basis of them length of edges, areas of face groups resulting from further division and areas of individual faces of polyhedra are calculated in this work. Furthermore, to each analyzed polyhedra, a semi-central view of ¼ of the grid was generated [1, 5] and the results were tabulated, for simplification – for 1/8 of each polyhedra.

The study was made for assumed angular and rectangular coordinate system.

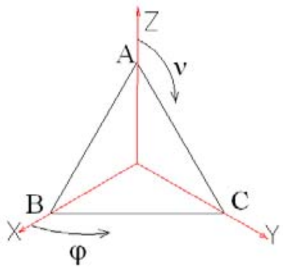

Fig. 3: Coordinate system assumed

Rectangular coordinates were calculated based on angular coordinates and on formulas [4]:

$$X = R * \cos\varphi * \sin\upsilon$$
$$Y = R * \sin\varphi * \sin\upsilon$$
$$Z = R * \cos\upsilon$$

where : R - radius

Table 2: Angular and rectangular coordinates of semi-regular 32-hedron

W	angular coord. φ	angular coord. υ	rectangular coord. X	rectangular coord. Y	rectangular coord. Z
	32-hedron				
01	0°	0°	0,000000	0,000000	1,000000
11	0°	45°	0,707107	0,000000	0,707107
12	90°	45°	0,000000	0,707107	0,707107
21	0°	90°	1,000000	0,000000	0,000000
22	45°	90°	0,707107	0,707107	0,000000
23	90°	90°	0,000000	1,000000	0,000000

W - vertices

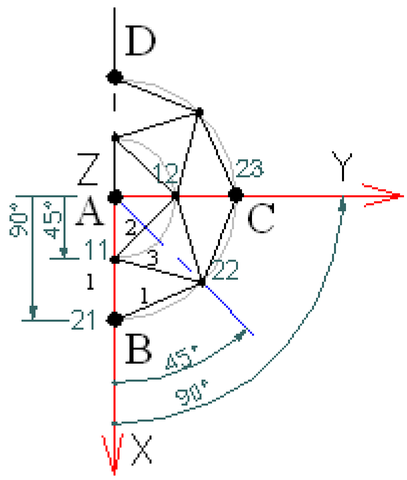

Fig. 4: A semi-central view of one half of the grid of a 32-hedron

Table 3: Angular and rectangular coordinates of semi-regular 72-hedron

72-hedron					
W	angular coord.		rectangular coord.		
	φ	υ	X	Y	Z
01	0°	0°	0,000000	0,000000	1,000000
11	0°	30°	0,500000	0,000000	0,866035
12	90°	30°	0,000000	0,500000	0,866025
21	0°	60°	0,866025	0,000000	0,500000
22	45°	60°	0,612372	0,612372	0,500000
23	90°	60°	0,000000	0,866025	0,500000
31	0°	90°	1,000000	0,000000	0,000000
32	30°	90°	0,866025	0,500000	0,000000
33	60°	90°	0,500000	0,866025	0,000000
34	90°	90°	0,000000	1,000000	0,000000

W - vertices

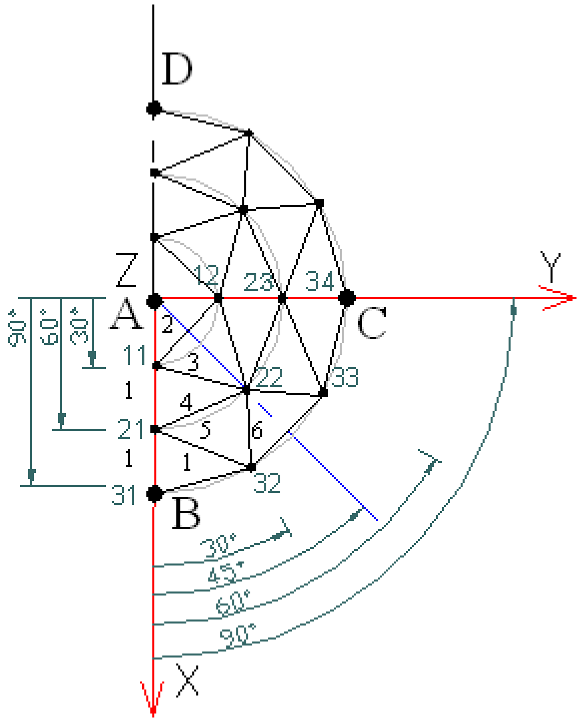

Fig. 5: A semi-central view of one half of the grid of a 72-hedron

Table 4: Angular and rectangular coordinates of semi-regular 128-hedron

128-hedron					
W	angular coord.		rectangular coord.		
	φ	υ	X	Y	Z
01	0°	0°	0,000000	0,000000	1,000000
11	0°	22°50`	0,382683	0,000000	0,923880
12	90°	22°50`	0,000000	0,382683	0,923880
21	0°	45°	0,707107	0,000000	0,707107
22	45°	45°	0,500000	0,500000	0,707107
23	90°	45°	0,000000	0,707107	0,707107
31	0°	67°50`	0,923880	0,000000	0,382683
32	30°	67°50`	0,800103	0,461940	0,382683
33	60°	67°50`	0,461940	0,800103	0,382683
34	90°	67°50`	0,000000	0,923880	0,382683
41	0°	90°	1,000000	0,000000	0,000000
42	22°50`	90°	0,923880	0,382683	0,000000
43	45°	90°	0,707107	0,707107	0,000000
44	67°50`	90°	0,382683	0,923880	0,000000
45	90°	90°	0,000000	1,000000	0,000000

W - vertices

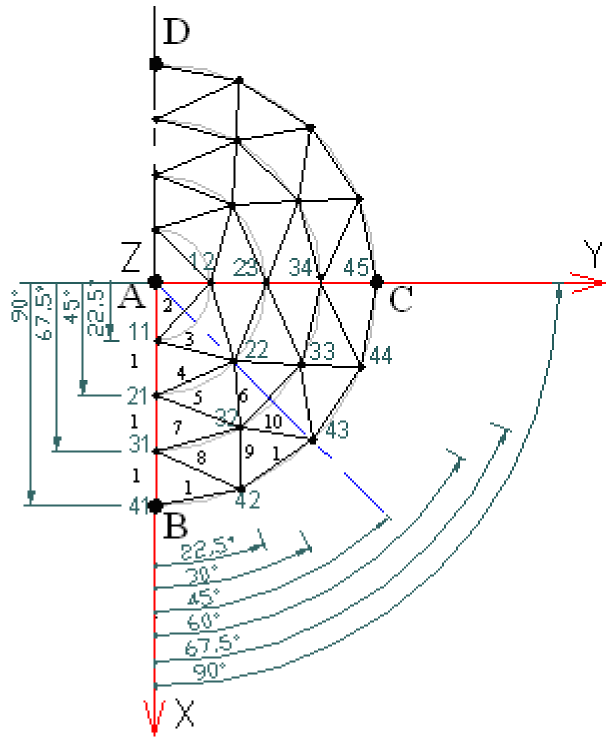

Fig. 6: A semi-central view of one half of the grid of a 128-hedron

Table 5: Angular and rectangular coordinates of semi-regular 200-hedron

200-hedron					
W	angular coord.		rectangular coord.		
	φ	υ	X	Y	Z
01	0°	0°	0,000000	0,000000	1,000000
11	0°	18°	0,309017	0,000000	0,951057
12	90°	18°	0,000000	0,309017	0,951057
21	0°	36°	0,587785	0,000000	0,809017
22	45°	36°	0,415627	0,415627	0,809017
23	90°	36°	0,000000	0,587785	0,809017
31	0°	54°	0,809017	0,000000	0,587785
32	30°	54°	0,700629	0,404509	0,587785
33	60°	54°	0,404509	0,700629	0,587785
34	90°	54°	0,000000	0,809017	0,587785

200-hedron, continued					
W	angular coord.		rectangular coord.		
	φ	υ	X	Y	Z
41	0°	72°	0,951057	0,000000	0,309017
42	22°50`	72°	0,878663	0,363953	0,309017
43	45°	72°	0,672499	0,672499	0,309017
44	67°50`	72°	0,363953	0,878663	0,309017
45	90°	72°	0,000000	0,951057	0,309017
51	0°	90°	1,000000	0,000000	0,000000
52	18°	90°	0,951057	0,309017	0,000000
53	36°	90°	0,809017	0,587785	0,000000
54	54°	90°	0,587785	0,809017	0,000000
55	72°	90°	0,309017	0,951057	0,000000
56	90°	90°	0,000000	1,000000	0,000000

W - vertices

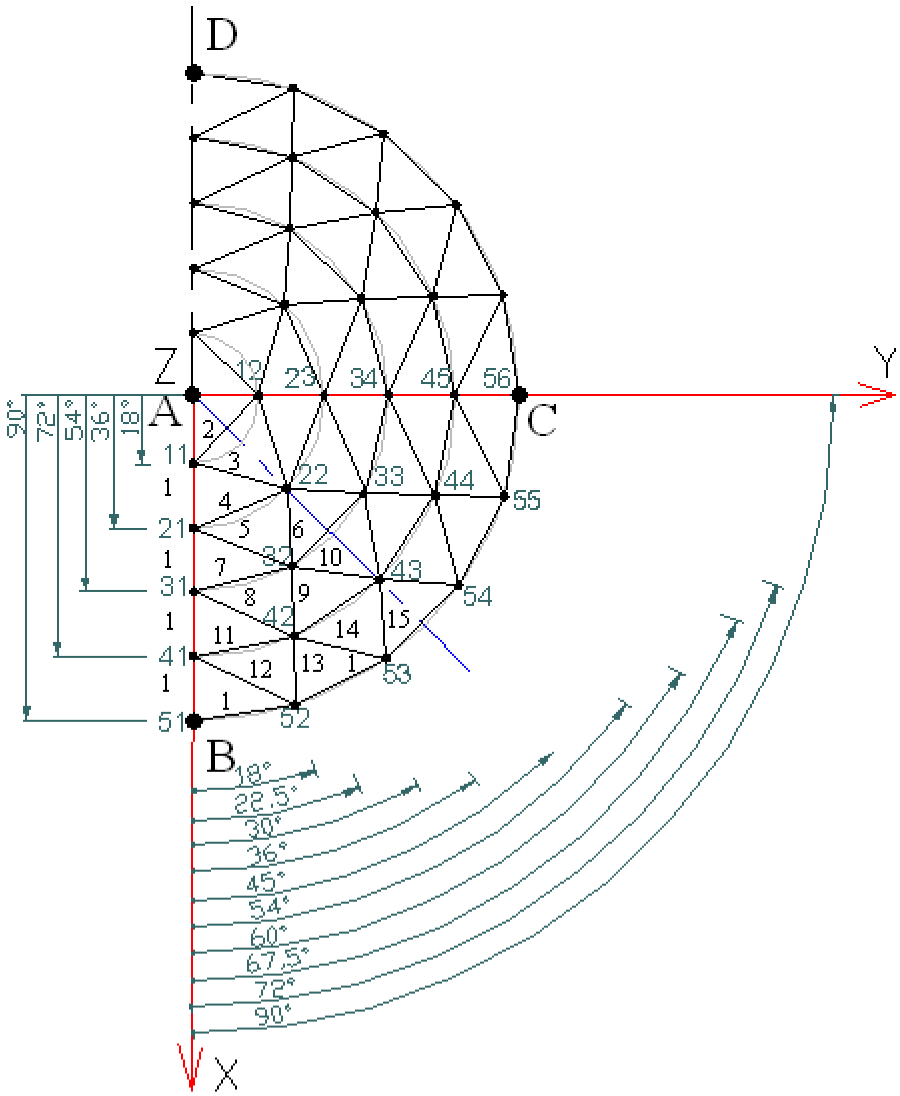

Fig. 7: A semi-central view of one half of the grid of a 200-hedron

Table 6: Angular and rectangular coordinates of semi-regular 288-hedron

288-hedron					
W	angular coord.		rectangular coord.		
	φ	υ	X	Y	Z
01	0°	0°	0,000000	0,000000	1,000000
11	0°	15°	0,258819	0,000000	0,969926
12	90°	15°	0,000000	0,258819	0,969926
21	0°	30°	0,500000	0,000000	0,866025
22	45°	30°	0,353554	0,353554	0,866025
23	90°	30°	0,000000	0,500000	0,866025
31	0°	45°	0,707107	0,000000	0,707107
32	30°	45°	0,612372	0,353554	0,707107
33	60°	45°	0,353554	0,612372	0,707107
34	90°	45°	0,000000	0,707107	0,707107
41	0°	60°	0,866025	0,000000	0,500000
42	22°50`	60°	0,800103	0,331413	0,500000
43	45°	60°	0,612372	0,612373	0,500000
44	67°50`	60°	0,331413	0,800103	0,500000
45	90°	60°	0,000000	0,866025	0,500000

288-hedron, continued					
W	angular coord.		rectangular coord.		
	φ	υ	X	Y	Z
51	0°	75°	0,969926	0,000000	0,258819
52	18°	75°	0,922455	0,299724	0,258819
53	36°	75°	0,784687	0,570108	0,258819
54	54°	75°	0,570108	0,784687	0,258819
55	72°	75°	0,299724	0,922455	0,258819
56	90°	75°	0,000000	0,969926	0,258819
61	0°	90°	1,000000	0,000000	0,000000
62	15°	90°	0,969926	0,258819	0,000000
63	30°	90°	0,866025	0,500000	0,000000
64	45°	90°	0,707107	0,707107	0,000000
65	60°	90°	0,500000	0,866025	0,000000
66	75°	90°	0,258819	0,969926	0,000000
67	90°	90°	0,000000	1,000000	0,000000

W - vertices

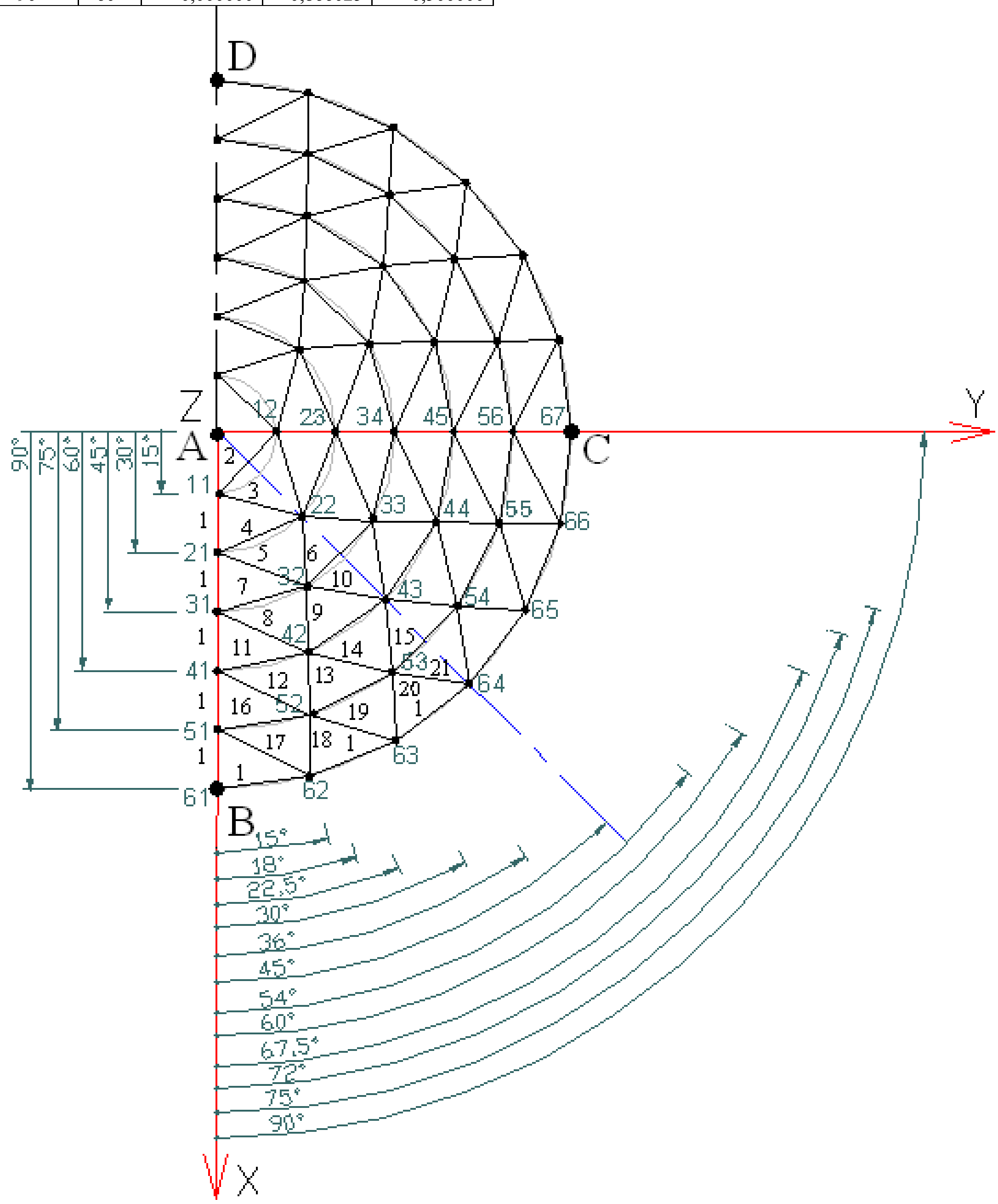

Fig. 8: A semi-central view of one half of the grid of a 288-hedron

Table 7: Angular and rectangular coordinates of semi-regular 392-hedron

392-hedron					
W	angular coord.		rectangular coord.		
	φ	υ	X	Y	Z
01	0°	0°	0,000000	0,000000	1,000000
11	0°	12°85`71,4``	0,222521	0,000000	0,974928
12	90°	12°85`71,4``	0,000000	0,222521	0,974928
21	0°	25°71`42,9``	0,433884	0,000000	0,900969
22	45°	25°71`42,9``	0,306802	0,306802	0,900969
23	90°	25°71`42,9``	0,000000	0,433884	0,900969
31	0°	38°57`14,3``	0,623490	0,000000	0,781831
32	30°	38°57`14,3``	0,539960	0,311745	0,781831
33	60°	38°57`14,3``	0,311745	0,539960	0,781831
34	90°	38°57`14,3``	0,000000	0,623490	0,781831
41	0°	51°42`85,7``	0,781831	0,000000	0,623490
42	22°50`	51°42`85,7``	0,722318	0,299193	0,623490
43	45°	51°42`85,7``	0,552838	0,552838	0,623490
44	67°50`	51°42`85,7``	0,299193	0,722318	0,623490
45	90°	51°42`85,7``	0,000000	0,781831	0,623490
51	0°	64°28`57,2``	0,900969	0,000000	0,433884
52	18°	64°28`57,2``	0,856873	0,278415	0,433884
53	36°	64°28`57,2``	0,728899	0,529576	0,433884
54	54°	64°28`57,2``	0,529576	0,728899	0,433884
55	72°	64°28`57,2``	0,278415	0,856873	0,433884
56	90°	64°28`57,2``	0,000000	0,900969	0,433884
61	0°	77°14`28,6``	0,974928	0,000000	0,222521
62	15°	77°14`28,6``	0,941708	0,252330	0,222521
63	30°	77°14`28,6``	0,844312	0,487464	0,222521
64	45°	77°14`28,6``	0,689378	0,689378	0,222521
65	60°	77°14`28,6``	0,487464	0,844312	0,222521
66	75°	77°14`28,6``	0,252330	0,941708	0,222521
67	90°	77°14`28,6``	0,000000	0,974928	0,222521
71	0°	90°	1,000000	0,000000	0,000000
72	12°85`71,4``	90°	0,974928	0,222521	0,000000
73	25°71`42,9``	90°	0,900969	0,433884	0,000000
74	38°57`14,3``	90°	0,781831	0,623490	0,000000
75	51°42`85,7``	90°	0,623490	0,781831	0,000000
76	64°28`57,2``	90°	0,433884	0,900969	0,000000
77	77°14`28,6``	90°	0,222521	0,974928	0,000000
78	90°	90°	0,000000	1,000000	0,000000

W - vertices

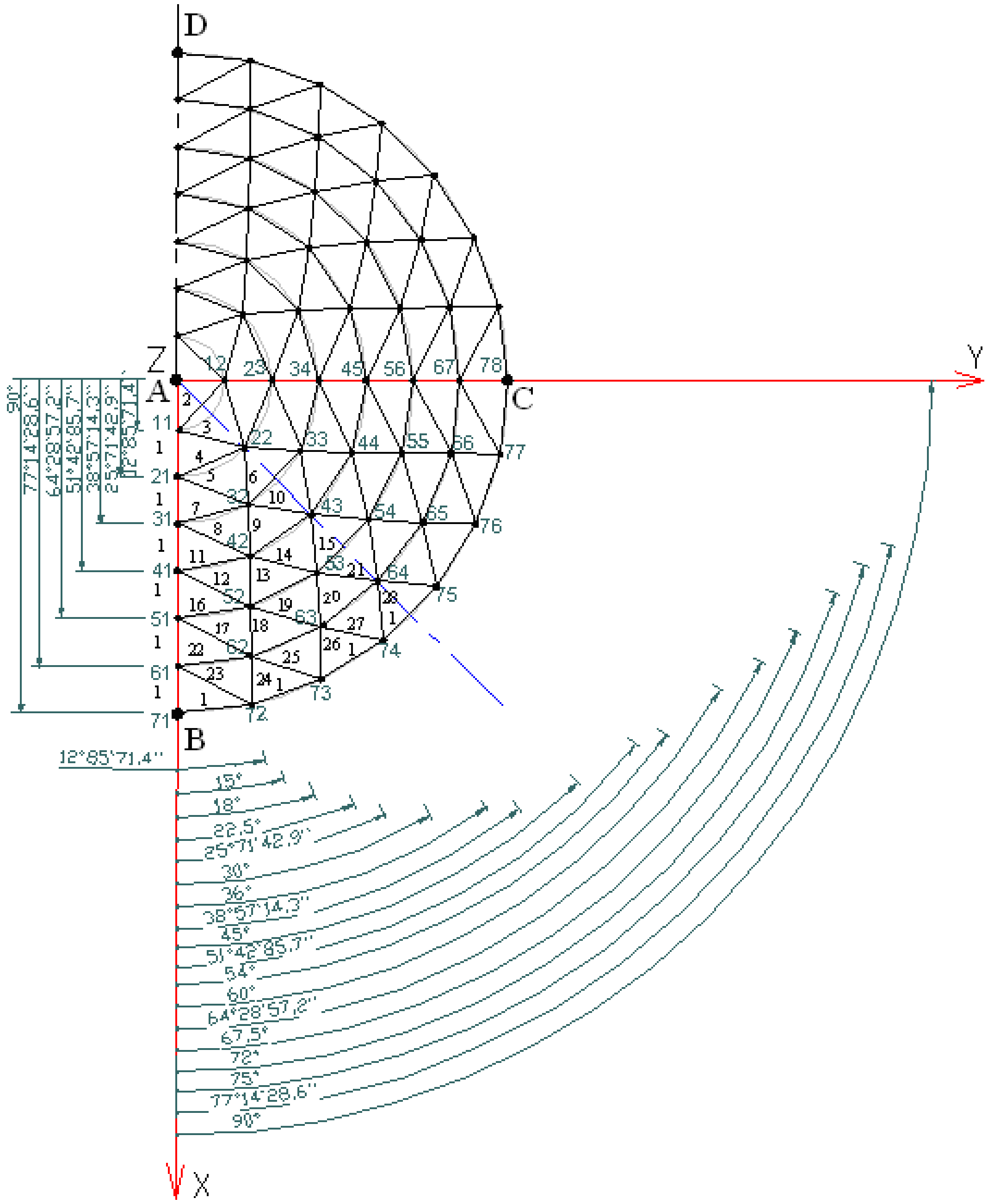

Fig. 9: A semi-central view of one half of the grid of a 392-hedron

3. The analysis of results

Using rectangular coordinates and formulas of distance of segment in space $\sqrt{(A_x - B_x)^2 + (A_y - B_y)^2 + (A_z - B_z)^2}$, length of edges for the part of 1/16 of octahedron-based polyhedra was calculated. These edges were marked in Fig. 4-9.

Table 8. Length of edges l_i of regular octahedron-based polyhedral

edge l_i	32-hedron	72-hedron	128-hedron	200-hedron	288-hedron	392-hedron	…
1	0,765367	0,517638	0,390180	0,312869	0,260560	0,223929	
2	1,000000	0,707107	0,541195	0,437016	0,366025	0,314692	
3	1,000000	0,722225	0,557453	0,451981	0,380487	0,326651	
4		0,662827	0,541196	0,449871	0,382684	0,332080	
5		0,707107	0,572091	0,474663	0,403587	0,350187	
6		0,571811	0,443578	0,360962	0,303713	0,261880	
7			0,478236	0,418779	0,366026	0,322742	
8			0,541195	0,463707	0,401716	0,352640	
9			0,409937	0,333245	0,280404	0,241834	
10			0,463898	0,387713	0,331482	0,288727	
11				0,371083	0,337906	0,305055	
12				0,437016	0,388828	0,345104	
13				0,322103	0,272291	0,233425	
14				0,387870	0,339678	0,298446	
15				0,348289	0,299411	0,259787	
16					0,303460	0,281885	
17					0,366025	0,331669	
18					0,266297	0,229242	
19					0,332087	0,297546	
20					0,280211	0,244475	
21					0,302943	0,267905	
22						0,254507	
23						0,314692	
24						0,226953	
25						0,290064	
26						0,235789	
27						0,268184	
28						0,249809	
…							…

On the basis of data from Table 8, the number of different edges groups and the number of groups with different area faces for one face of further polyhedra were calculated. The number is the same.

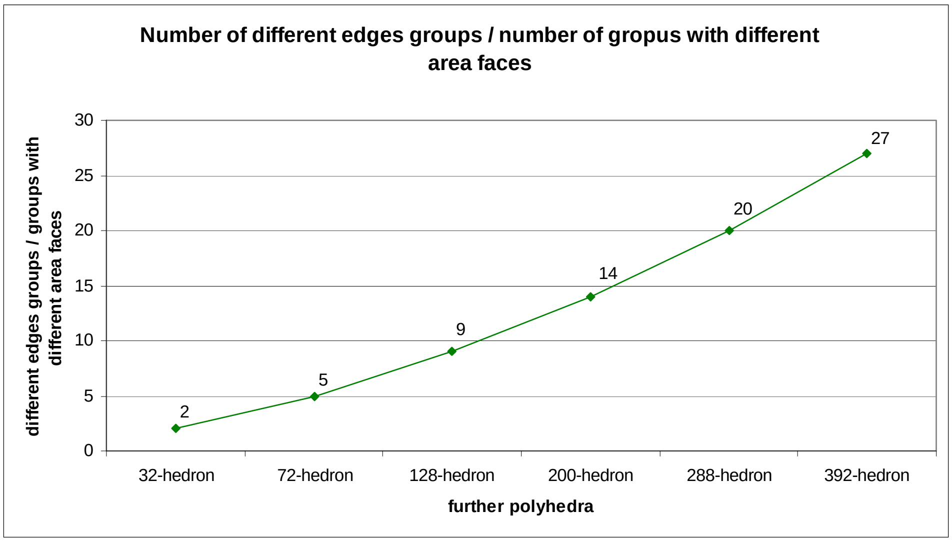

Fig. 10: Number of different edges groups and number of groups with different area faces

Rectangular coordinates were also used to calculate areas of separate polyhedra. The algorithm was generated in MsExcel program in the following way:

a) angular coordinates were calculated,
b) on the basis of angular coordinates, rectangular coordinates were calculated,
c) rectangular coordinates were used to calculation of area of one face of polyhedra, using the following formula:

$$S^2 = \frac{1}{4}\left\{ \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2 + \begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix}^2 + \begin{vmatrix} z_1 & x_1 & 1 \\ z_2 & x_2 & 1 \\ z_3 & x_3 & 1 \end{vmatrix}^2 \right\}$$

$$\begin{aligned} S = \sqrt{\begin{aligned} &\tfrac{1}{4}(((A_x \cdot B_y)+(A_y \cdot C_x)+(B_x \cdot C_y)-(B_y \cdot C_x)-(A_x \cdot C_y)-(A_y \cdot B_x))^2 + \\ &+((A_y \cdot B_z)+(A_z \cdot C_y)+(B_y \cdot C_z)-(B_z \cdot C_y)-(A_y \cdot C_z)-(A_z \cdot B_y))^2 + \\ &+((A_z \cdot B_x)+(A_x \cdot C_z)+(B_z \cdot C_x)-(B_x \cdot C_z)-(A_z \cdot C_x)-(A_x \cdot B_z))^2) \end{aligned}} \end{aligned}$$

d) areas of faces based on one face of regular octahedron were multiplied by 8, obtaining the area of whole polyhedra (Fig. 11).

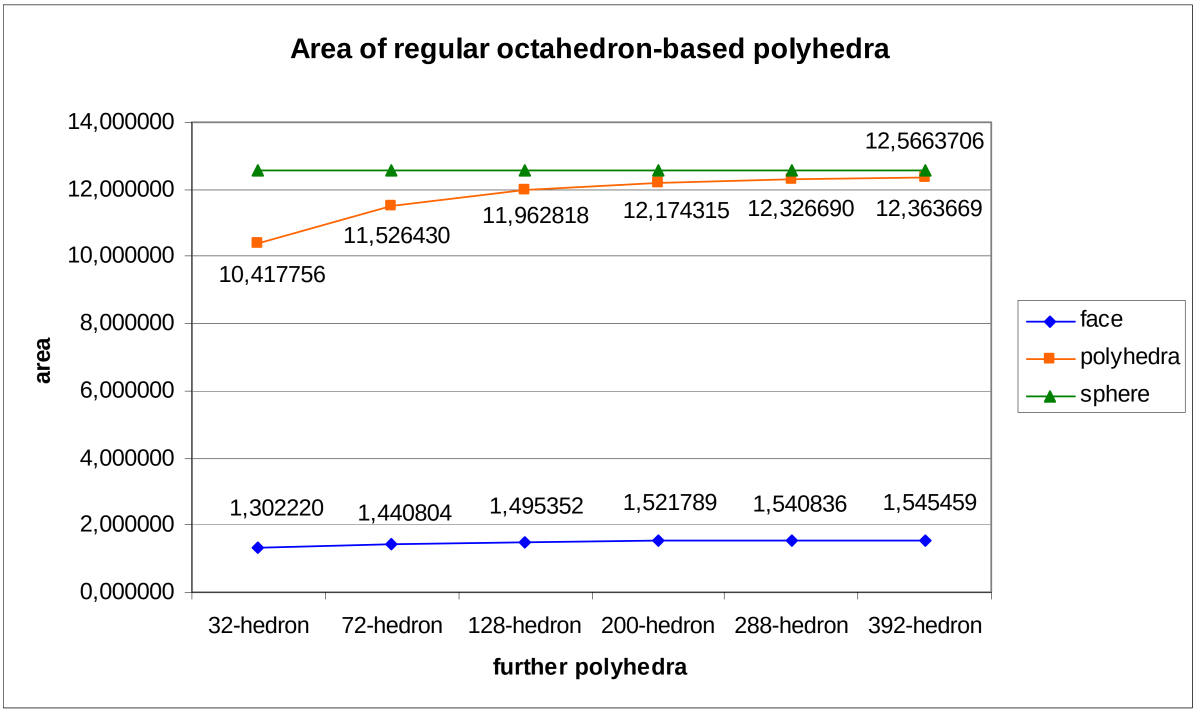

Fig. 11: Area of regular octahedron-based polyhedra

Table 9 shows formulas used for calculation of number of edges, nodes as well as different edges groups and different area faces groups.

Table 9: Formulas for number of edges, vertices as well as different edges groups and groups with different area faces calculation

polyhedra	number of edges	number of vertices	number of different edges groups number of different area faces groups
	$p_w * n^2$	$0,5 * N * n^2 + 2$	**2+m**
8-hedron	12	6	2
32-hedron	48	18	5
72-hedron	108	38	9
128-hedron	192	66	14
200-hedron	300	102	20
288-hedron	432	146	27
392-hedron	588	198	35
512-hedron	768	258	44
648-hedron	972	326	54
…	…	…	…

where:
p_w – the number of edges of octahedron,
n – consecutive natural numbers,
N – the number of face of octahedron,
m – consecutive natural numbers, starting from number 3, adding to the previous result.

4. Conclusion

The topology and calculation methods (using algorithms generated in MsExcel program) of geometric parameters of octahedron-based polyhedra (with the bigger number of faces) were presented in this study. Concentrating on six polyhedra is enough for preparing conclusion. It is possible to apply the analysis of the next polyhedra to the calculation methodology.

On the basis of analyzed six polyhedra, it is possible to generate formula for the number of different edges groups which are identical with the number of groups with different area faces (Fig. 10) for each polyhedra from diagonally composition of Table 1: **2 + m**. The results are shown in Table 9.

The way of topology and geometry creation of octahedron-based polyhedra with nodes lying on the sphere, presented in this paper, is the base of choice and analysis of frame structure for designers. In comparison to western countries (mainly USA) frame structure in Poland are still relatively seldom used.

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SIATKI SFERYCZNE POWSTAŁE NA BAZIE OŚMIOŚCIANU FOREMNEGO

Praca jest poświęcona obliczeniom parametrów geometrycznych wielościanów o większej liczbie ścian – pochodnych od ośmiościanu foremnego. Algorytmy utworzono w programie MsEXCEl. Skoncentrowano się na trzech wielościanach pochodnych o parzystej liczbie oczek oraz na trzech o nieparzystej liczbie oczek. Na tej podstawie sformułowano wnioski końcowe.