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THE CONIC OF CENTERS S^2 OF A PENCIL $P^2_{1=2=3,4}$

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Abstract. The **E**-transformation is quadratic in the projective 2-dimensional space and based on the circle $\mathbf{n^2}$ and the center **W**, which lies on the circle $\mathbf{n^2}$. In the **E**-transformation to the straight line **a'** corresponds a conic $\mathbf{a^2}$. The elation has been defined, where **a'** is a vanishing line, the line $\mathbf{t_a}$ parallel to **a'** and passing through the point **W** is the axis of elation. All lines that do not pass through the center of the transformation **W** will correspond to osculary conics passing through the three points 1=2=3 coinciding with the center **W**. The centers of these conics make also a conic of centers s^2. Special cases are distinguished dependent on whether the base quadrangle 1=2=3,4 is concave or convex. The case with point 4 lying at infinity has been discussed. Two theorems have been formulated and proved.

Keywords. Projective geometry, conic of centers, base quadrangle, pencils of osculary tangent conics, elation.

1. Definition of the E-transformation

The **E**-transformation is a quadratic transformation defined as follows. Let us distinguish the point **W**, which lies in the given circle $\mathbf{n^2}$. We determine the tangent **w** to the circle $\mathbf{n^2}$ at point **W**. (Fig.1). Let us assume that the arbitrary line **a'** be the vanishing line of the elation defined with the center **W** and the axis of elation $\mathbf{t_a}$, which is parallel to the line **a'** and passes through point **W**. We call a perspective collineation an elation according as the center and axis are incident (Coxeter [1], p.248), which is here the case. Let us determine the relation between the arbitrary line **a'** and the conic $\mathbf{a^2}$. We assume that the circle $\mathbf{n^2}$ will be transformed in the defined elation into the conic $\mathbf{a^2}$. The correspondence between the line **a'** and the conic a^2 will be called a quadratic **E**-transformation. Certainly, the type of the conic $\mathbf{a^2}$ received in the **E**-transformation depends on the mutual position of the vanishing line **a'** and the circle $\mathbf{n^2}$.

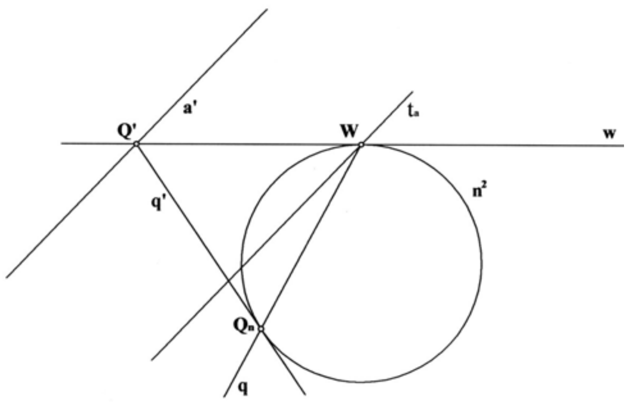

Fig. 1: The basis of the E transformation

In the publication (Wojtowicz [6]) special cases of the line **a'** and the circle $\mathbf{n^2}$ mutual position have been discussed. The following cases are distinguished depending on whether the line **a'** and the circle $\mathbf{n^2}$ are disjoint or not.

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1) In case the line **a’** is external in reference to the circle $\mathbf{n^2}$, then the corresponding conic $\mathbf{a^2}$ is an **ellipse**.
2) In case the line **a’** is tangent to the circle $\mathbf{n^2}$, then the corresponding conic $\mathbf{a^2}$ is a **parabola**.
3) In case the line **a’** is the secant of the circle $\mathbf{n^2}$, then the corresponding conic $\mathbf{a^2}$ is a **hyperbola**.

Let us now determine the basic elements of the conic $\mathbf{a^2}$, which corresponds to the line **a’**. In order to draw the diameter of the conic $\mathbf{a^2}$ we consider the point **Q’** at which lines **w** and **a’** meet. The polar **q** of the point **Q’** with respect to the circle $\mathbf{n^2}$ will be transformed into one diameter of the conic $\mathbf{a^2}$ (Fig.1).

2. Special cases of the E-transformation

It has been proved (Wojtowicz [6]) that the lines **a’**, **b’**, **c’**, … not passing through the point **W** will be transformed into osculary tangent conics $\mathbf{a^2}$, $\mathbf{b^2}$, $\mathbf{c^2}$,…. . The three coinciding points of tangency **1=2=3** of these conics coincide with the point **W**. Point **4** can be optionally chosen on one of the conics $\mathbf{a^2}$, $\mathbf{b^2}$, $\mathbf{c^2}$,…. .

Let us now consider some particular cases depending on the position of the base points **1=2=3,4** of the pencil $\mathbf{P^2}_{\mathbf{1=2=3,4}}$ of osculary tangent conics $\mathbf{a^2}$, $\mathbf{b^2}$, $\mathbf{c^2}$**,….** . If point **4** lies on the same side with respect to the line **w** as the circle $\mathbf{n^2}$, then we can consider the base quadrangle of the pencil to be convex, if point **4** lies on the opposite side with respect to the line **w** to the circle $\mathbf{n^2}$, then the quadrangle **1=2=3,4** is concave (Wojtowicz [6], p.7-10).

The centers of the conics, which create the pencil $\mathbf{P^2}_{\mathbf{1=2=3,4}}$, lie on the conic called $\mathbf{s^2}$ (Plamitzer [5], p. 61-63).

We can distinguish the following particular cases.

CASE I. If the base quadrangle **1=2=3, 4** is convex then the conic $\mathbf{s^2}$ is a hyperbola (Fig.2).
CASE II. If the base quadrangle **1=2=3, 4** is concave then the conic $\mathbf{s^2}$ is an ellipse.
CASE III. If the fourth vertex of the base quadrangle **1=2=3,** $\mathbf{4^\infty}$ lies at infinity then the conic $\mathbf{s^2}$ is a hyperbola.

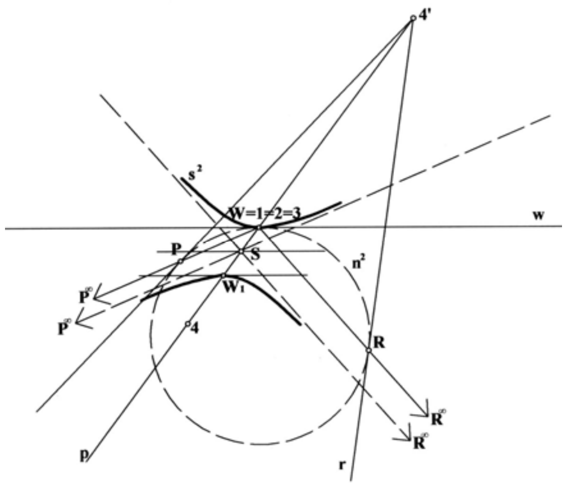

Fig. 2: The conic of centers s^2 is a hyperbola

Let us discuss the case with a hyperbola $\mathbf{s^2}$ (CASE I). It has been assumed that the fourth vertex **4** of the quadrangle lies inside the circle $\mathbf{n^2}$ and the quadrangle **1=2=3,4** is convex (Fig.2). Thus the conic $\mathbf{s^2}$ is a hyperbola. The segment $\overline{WW_1}$ (point $\mathbf{W_1}$ is a midpoint of the segment $\overline{W4}$) is the diameter of the constructed hyperbola. The midpoint **S** of the segment

$\overline{WW_1}$, is a center of this hyperbola. The points **4'**, **P** and **R** (respectively the tangent points of the lines **p** and **r** drawn from the point **4'** to the circle $\mathbf{n^2}$) have been also determined. To points **P** and **R** correspond respectively points $\mathbf{P^\infty}$ and $\mathbf{R^\infty}$, which are the centers of the two parabolas, which belong to the pencil $P^2{}_{1=2=3,4}$ of conics. The asymptotes of the hyperbola are passing through the point **S** and the two points $\mathbf{P^\infty}$ and $\mathbf{R^\infty}$.

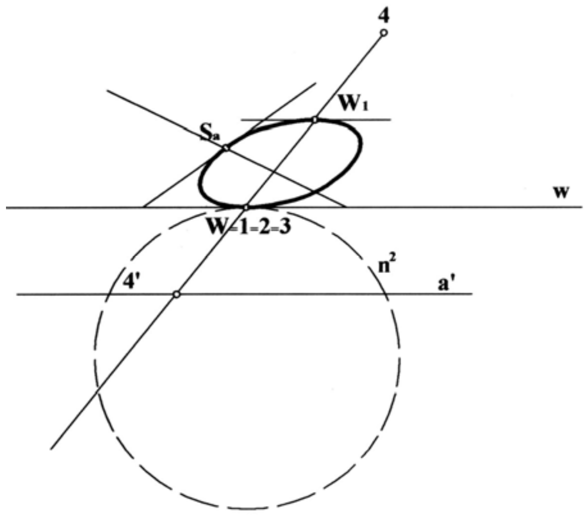

Fig. 3: An ellipse as a case of the conic of centers $\mathbf{s^2}$

The base quadrangle **1=2=3, 4** presented in Fig. 3 is convex as the point **4** lies on the opposite side to the circle $\mathbf{n^2}$ in respect to the line **w**. Thus the conic of the centers is an ellipse. The diameter WW_1 of the conic $\mathbf{s^2}$ and two conjugate with this diameter tangents **w** and $\mathbf{w_1}$ are determined.

In Fig. 4 a special case of the base quadrangle **1=2=3,** $\mathbf{4^\infty}$ is presented. Point $\mathbf{4^\infty}$ corresponds to the point **4'** lying on the circle $\mathbf{n^2}$. The line **a'** parallel to the line **w** and passing through the point **4'** has been specified. The center $\mathbf{S_a}$ of the hyperbola $\mathbf{a^2}$ corresponding to the line **a'** is determined. The conic of centers $\mathbf{s^2}$ is a parabola in this case. This parabola will be defined with the diameter $\mathbf{W4^\infty}$ conjugate to the tangent **w** at the point **W** and the point $\mathbf{S_a}$.

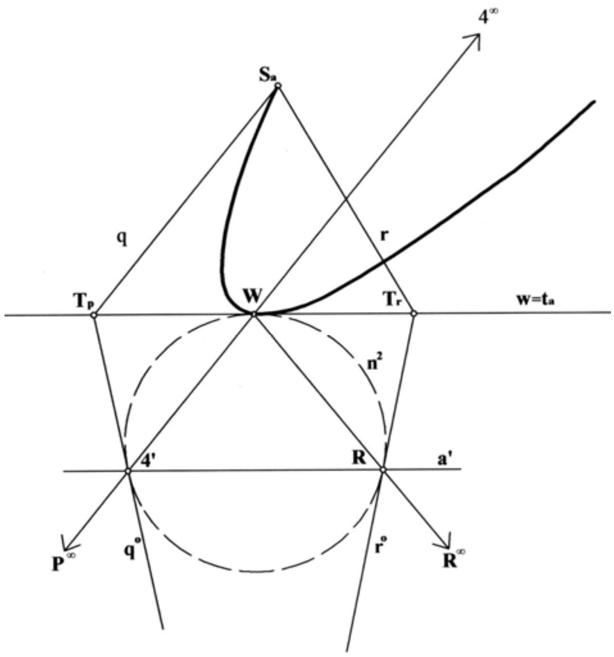

Fig. 4: For the base quadrangle **1=2=3,** $\mathbf{4^\infty}$ a parabola will be the conic of centers $\mathbf{s^2}$

Theorem 1: The pencil of conics $\mathbf{P^2_{1=2=3,4}}$, to which belong conics $\mathbf{a^2}$, $\mathbf{b^2}$, $\mathbf{c^2}$,…., is in projective relation to the range of points of the second order with the base on the conic of centers $\mathbf{s^2}$ and the elements $\mathbf{S_a}$, $\mathbf{S_b}$, $\mathbf{S_c}$,…. being respectively the centers of the conics $\mathbf{a^2}$, $\mathbf{b^2}$, $\mathbf{c^2}$,….

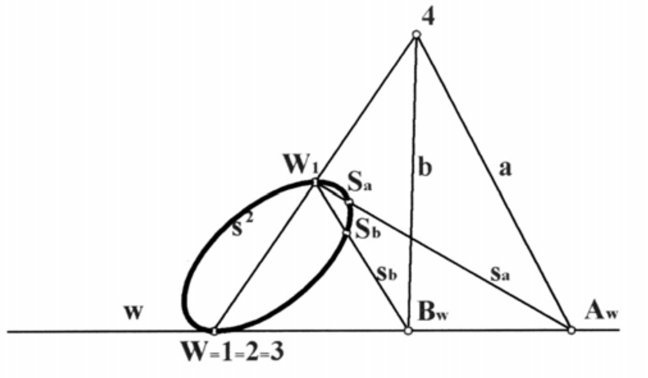

Fig. 5: Projective relation between the pencil of conics $\mathbf{P^2_{1=2=3,4}}$ and the range $\mathbf{s^2}$

Proof. Let us determine point **4** and draw an optional line **a** passing through this point (Fig.5). Line **a**, which is a tangent to the conic at point **4** together with the points **1=2=3** defines the conic $\mathbf{a^2}$ of the pencil $\mathbf{P^2_{1=2=3,4}}$. The line **a** meets the line **w** at point $\mathbf{A_w}$, which is a pole of the secant $\overline{W4}$ of the conic $\mathbf{a^2}$. Thus the line $\mathbf{s_a = A_w W_1}$, where $\mathbf{W_1}$ is the midpoint of the segment $\overline{W4}$, is the diameter of the conic $\mathbf{a^2}$ and it meets the conic $\mathbf{s^2}$ at the point $\mathbf{S_a}$, which is the center of the conic $\mathbf{a^2}$. Similarly, we construct the line $\mathbf{s_b}$ and the center $\mathbf{S_b}$ of another conic $\mathbf{b^2}$. In consequence we obtain the following range of perspective elements:

$$\mathbf{P^2_{1=2=3,4}\,(a^2, b^2, c^2, \ldots.)} \;\overline{\barwedge}\; \mathbf{4(a, b, c, \ldots)} \;\overline{\barwedge}\; \mathbf{w(A_w, B_w, C_w, \ldots)} \;\overline{\barwedge}$$
$$\overline{\barwedge}\; \mathbf{W_1(s_a, s_b, s_c, \ldots)} \;\overline{\barwedge}\; \mathbf{s^2(S_a, S_b, S_c, \ldots)}$$

The boundary elements of this projective chain are projective

$$\mathbf{P^2_{1=2=3,4}\,(a^2, b^2, c^2, \ldots.)} \;\barwedge\; \mathbf{s^2(S_a, S_b, S_c, \ldots)}$$

as stated.

3. Properties of the E-transformation

Theorem 2: The radius of the osculary tangent circle $\mathbf{n_1^2}$ at point **W** of the conic of centers $\mathbf{s^2}$ is half the length of the radius of the circle $\mathbf{a^2}$.

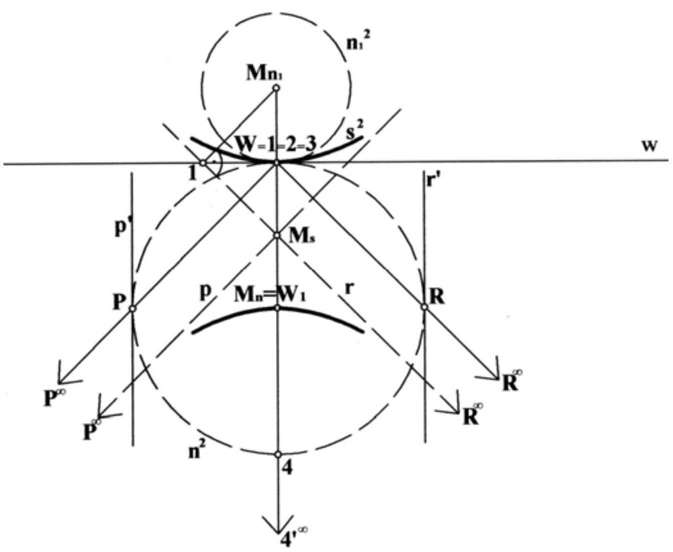

Fig. 6: Construction of the osculary tangent circle n_1^2 to the conic of centers s^2

The proof for the Theorem 2 will be presented for the case of a hyperbola.

Proof. Let the circle $\mathbf{n}^2$ be given together with the point **W=1=2=3** belonging to this circle. We then specify point **4** lying on the circle $\mathbf{s}^2$ in the opposite position with respect to the point **W** (Fig.6).

The center of the hyperbola of centers $\mathbf{s}^2$ lies at a half distance of the length of the circle's $\mathbf{n}^2$ radius. The asymptotes **p** and **r** make the 45° angle with respect to the line **w**. We now consider the lines **w** and **r**, which meet at point **1**. Let us draw a perpendicular line to the asymptote **r** from point **1**. This perpendicular meets the axis $\overline{WW_1}$ at point $\mathbf{M_{n1}}$, which is the center of the osculary tangent circle to the conic $\mathbf{s}^2$ at point **W**. The segments $\overline{WM_{n1}}$ and $\overline{WM_s}$ are equal length and thus the radius of the circle $\mathbf{n_1}^2$ is half the length of the radius of the circle $\mathbf{n}^2$, as stated.

From the Theorem 2 it follows that when the conic $\mathbf{s}^2$ is given, then it is an easy task to determine the base circle $\mathbf{n}^2$ of the **E**-transformation and, in consequence, to determine conjugate diameters or the asymptotes of the conics belonging to the pencil $\mathbf{P}^2{}_{1=2=3,4}$ (Kaczmarek [7]).

In Fig.7 the hyperbola $\mathbf{s}^2$ is given to be the conic of the centers of the pencil $\mathbf{P}^2{}_{1=2=3,4}$. The circle $\mathbf{n_1}^2$ corresponding to the hyperbola $\mathbf{s}^2$ and the circle $\mathbf{n}^2$ as the base of the **E**-transformation have been determined. An optionally chosen point $\mathbf{M_a}$ on the hyperbola $\mathbf{s}^2$ is the center of the conic $\mathbf{a}^2$ to be constructed. The tangent $\mathbf{s_a}$ to this conic at point **4** has been constructed.

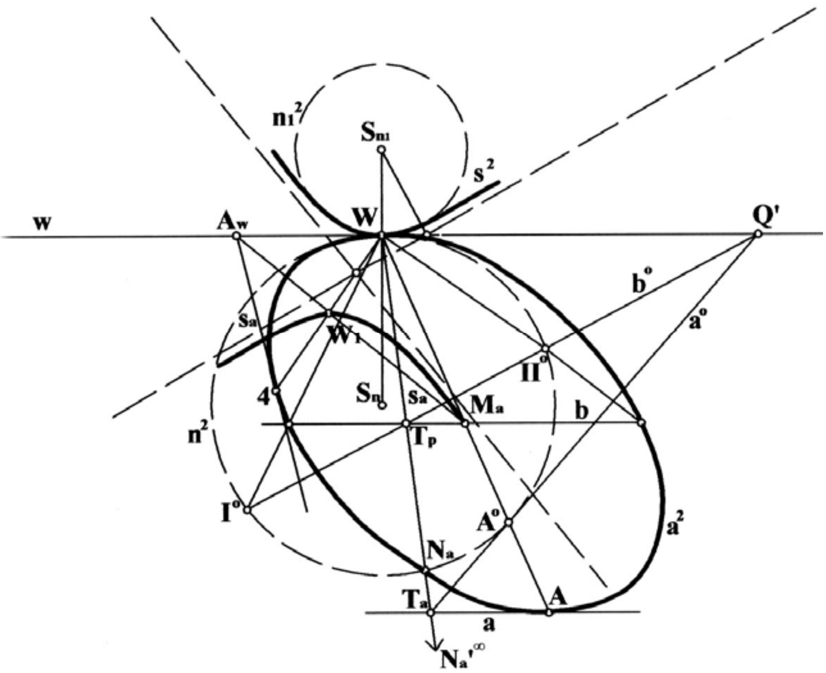

Fig. 7: Hyperbola as a conic of the centers $\mathbf{s}^2$ and the construction of an exemplary ellipse with the center $\mathbf{M_a}$

The proof for the Theorem 2 will be similar for the other types of conics. Fig.8 illustrates the case, where the conic of centers $\mathbf{s}^2$ is assumed to be a circle. The base circle $\mathbf{n}^2$ of the **E**-transformation is double size of the circle $\mathbf{s}^2$, while the point **4'** coincides with the center of the circle $\mathbf{n}^2$.

Let us choose an arbitrary point $\mathbf{M_a}$ on the circle $\mathbf{s}^2$ to be a center of a hyperbola $\mathbf{a}^2$. The asymptotes of the hyperbola and the tangent **ā** at point **4** are constructed.

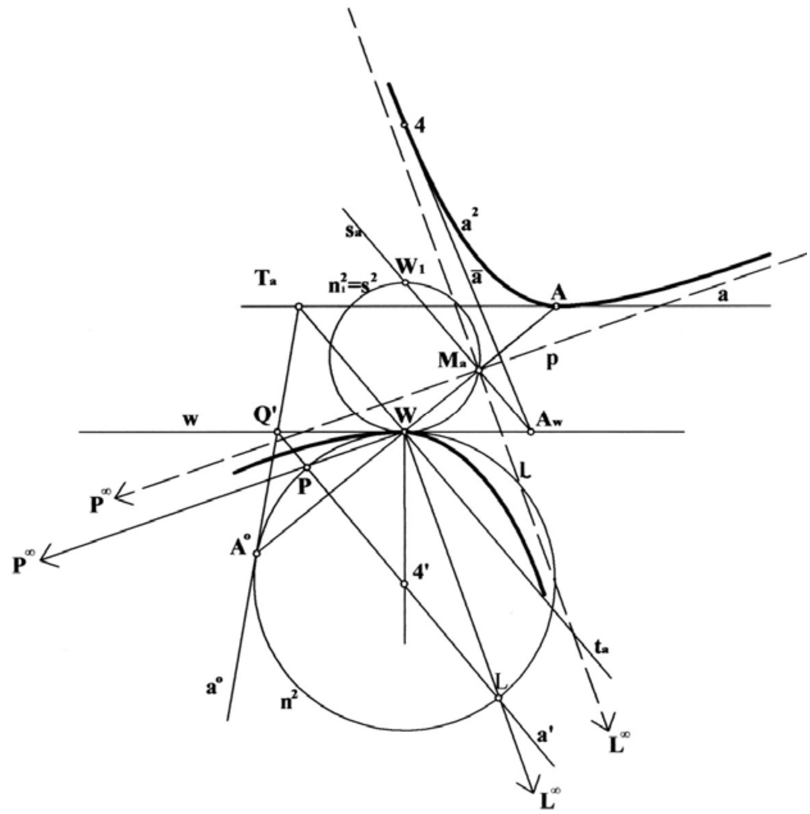

Fig. 8: Special case of the **E**-transformation where the conic $\mathbf{s^2}$ coincides with the conic $\mathbf{n_1{}^2}$

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STOŻKOWE ŚRODKÓW PĘKU $\mathbf{P^2{}_{1=2=3,4}}$

Praca jest kontynuacją artykułu „Pęki stożkowych nadściśle stycznych ($\mathbf{P^2{}_{1=2=3,4}}$)” ([6]), w której omówiono przekształcenie kwadratowe „**E**”, dla którego bazą jest okrąg $\mathbf{n^2}$, natomiast środkiem przekształcenia jest punkt **W** leżący na okręgu $\mathbf{n^2}$. Stwierdzono, że wszystkie proste, które nie przechodzą przez punkt **W**, przekształcają się w stożkowe wzajemnie ściśle styczne czyli przechodzące przez trzy punkty **1=2=3** pokrywające się z punktem **W**. Środki poszczególnych stożkowych pęku leżą na stożkowej, którą nazwano stożkowa środków i oznaczono $\mathbf{s^2}$. W pracy omówiono trzy przypadki, w których w zależności od czworokąta podstawowego **1=2=3,4** stożkowa środków $\mathbf{s^2}$ jest hiperbolą, elipsą, parabolą. Przedstawiono również twierdzenie, z którego wynika, iż mając zadaną stożkową środków $\mathbf{s^2}$ można wyznaczyć bazę $\mathbf{n^2}$ przekształcenia „**E**" oraz wyznaczyć średnice sprzężone lub asymptoty poszczególnych stożkowych pęku $P^2{}_{1=2=3,4}$. W pracy pokazano, że pęk stożkowych $P^2{}_{1=2=3,4}$, którego elementami są stożkowe a^2, b^2, c^2,…. jest rzutowy do szeregu punktów rzędu drugiego, którego podstawą jest „stożkowa środków" s^2, a elementami są punkty S_a, S_b, S_c, ... będące środkami stożkowych a^2, b^2, c^2,…..