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PENCIL OF OSCULARY TANGENT CONICS $P^2_{1=2=3,4}$

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Abstract. In the paper a definition of a special quadratic transformation has been given. In the transformation conic **a^2**, which is an elation of a circle **n^2**, corresponds to an optional line **a'**.
Three special cases of the line **a'** layout in relation to the circle **n^2** have been considered. It has been proved that for all straight lines not passing through the center of elation **w**, are transformed into conics, which are all osculary tangent.
There has been formulated a new theorem considering the pencil of straight lines **4'(a',b',c'...)** and corresponding to this pencil, in transformation , a pencil of conics $P^2_{1=2=3,4}$ (**a^2,b^2,c^2...**). The theorem focuses on the projective property of the discussed two corresponding pencils.

Keywords: pencil of conics, conics.

1. Quadratic transformation

Let circle **n^2** be given as a basis of transformation with point **W** coinciding with this circle (Figure 1). Let point **W** be a center of transformation .

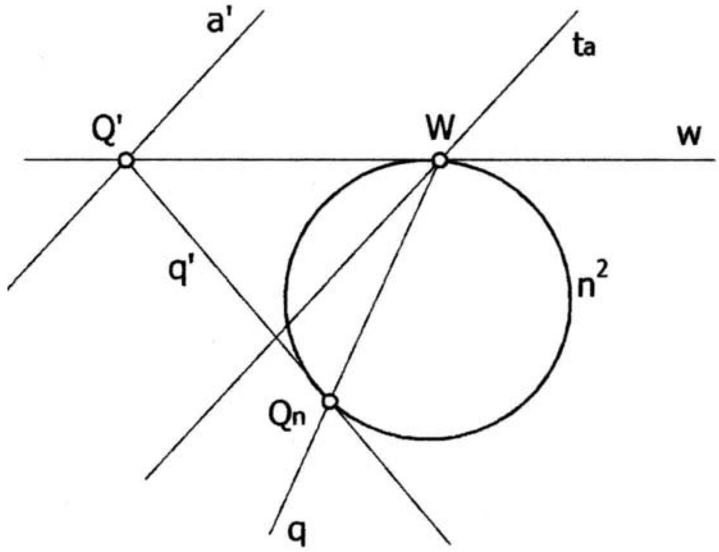

Figure 1.

2. Definition of transformation

To an optional line **a'** corresponds such a conic **a^2**, which is an elation of the circle **n^2**, where point **W** is a center and straight line **a'** is a vanishing line of this elation with respect to the circle configuration.

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The elation has been completed as the straight line $\mathbf{t_a}$ passing through point **W** and parallel to line **a**’ is an axis of this elation.

To determine the axis of a conic $\mathbf{a^2}$, so that it passes through point **W**, it is necessary to consider point **Q’**, where straight line **a’** intersects with the straight line **w** (which is a tangent to the circle $\mathbf{n^2}$ at point **W**) and to determine its polar line **q** with respect to the circle $\mathbf{n^2}$ (Figure 1).

In Figures 2a, 2b and 2c the straight line under consideration has been respectively chosen as: an external to a circle $\mathbf{n^2}$ line **a’** together with a corresponding ellipse $\mathbf{a^2}$; line **b’** tangent to $\mathbf{n^2}$ with a corresponding parabola $\mathbf{b^2}$, and finally line **c’**, which is a secant of the circle $\mathbf{n^2}$ together with a corresponding to hyperbola $\mathbf{c^2}$.

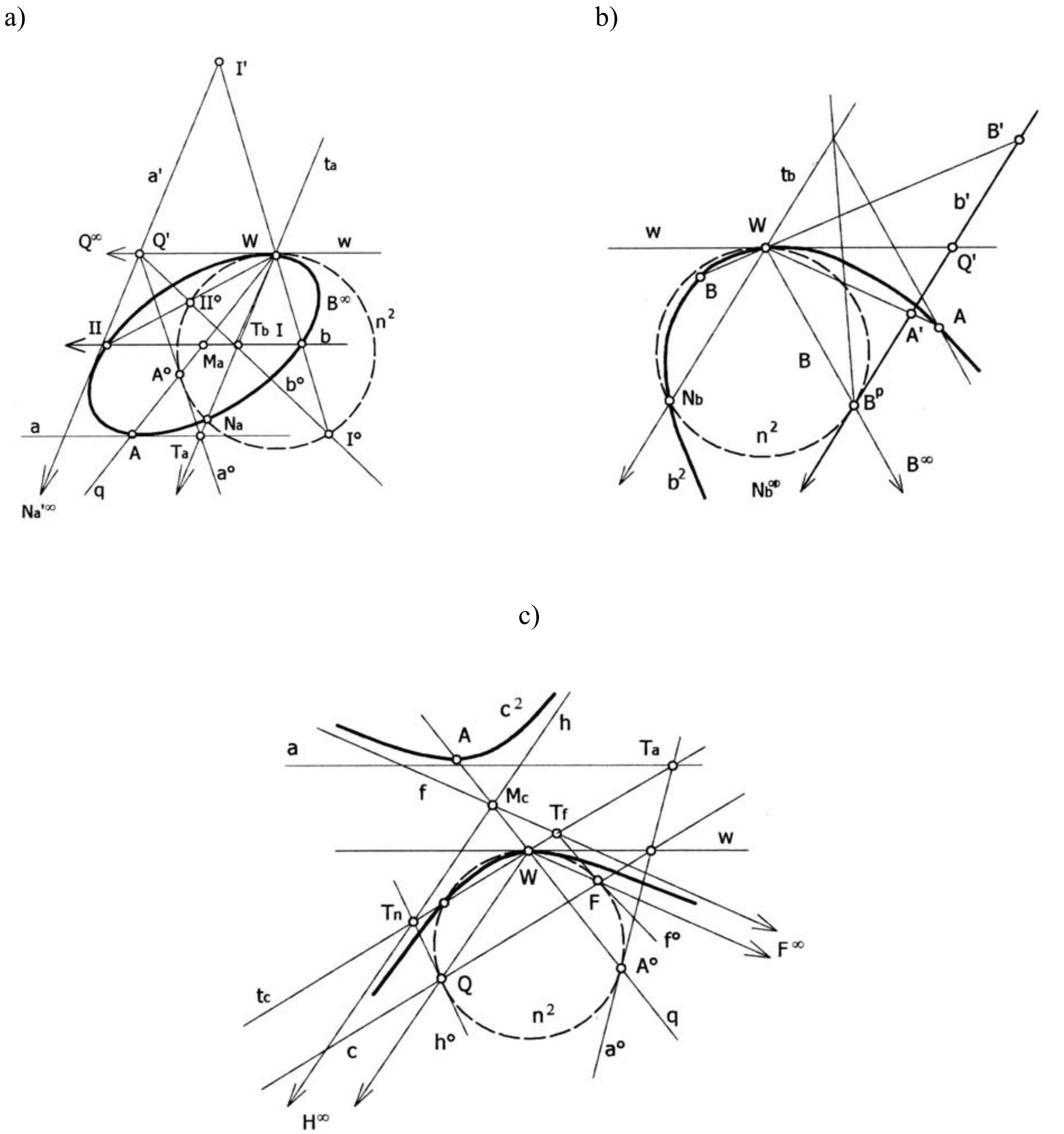

Figure 2.

Let us now notice that points at infinity $\mathbf{N_a}'^{\infty}$, $\mathbf{N_b}'^{\infty}$, $\mathbf{N_c}'^{\infty}$, coinciding respectively with lines **a'**, **b'**, **c'** , correspond to points $\mathbf{N_a}$, $\mathbf{N_b}$, $\mathbf{N_c}$ on the circle $\mathbf{n^2}$. Thus we may conclude that in transformation line at infinity $\mathbf{n}'^{\infty}$ corresponds to the basis of this transformation, namely to the circle $\mathbf{n^2}$.

Conics $\mathbf{a^2}$, $\mathbf{b^2}$, $\mathbf{c^2}$ are tangent to the circle $\mathbf{n^2}$ at point **W** (it results from the fact that the center of elation **W** lies on the circle $\mathbf{n^2}$) and meet this circle at another point $\mathbf{N_a}$, $\mathbf{N_b}$, $\mathbf{N_c}$. From the previous it follows that the circle $\mathbf{n^2}$ is an osculary tangent circle at point **W** to each of the discussed conics.

We can also conclude that each straight line, which does not pass through the point **W** will be transformed into osculary tangent conics, passing through three coinciding points **1=2=3**, which unite at point **W**.

3. Pencils of osculary tangent conics

Let us now determine the circle $\mathbf{n^2}$ with a distinguished point **W** at which three base points of a pencil of conics coincide **W**=(**1=2=3**). Let us also determine an optional point **4**, which together with points **1**=**2**=**3**, will determine a pencil of osculary tangent conics $P^2_{1=2=3,4}$.

If point **4** lies on the same side as the circle $\mathbf{n^2}$ with respect to the line **w**, then the quadrangle **1=2=3**,**4** will be considered as being convex (Figure 3a). If it lies on the opposite side with respect to the line **w**, then we will consider the quadrangle **1=2=3**,**4** to be concave (Figure 3b).

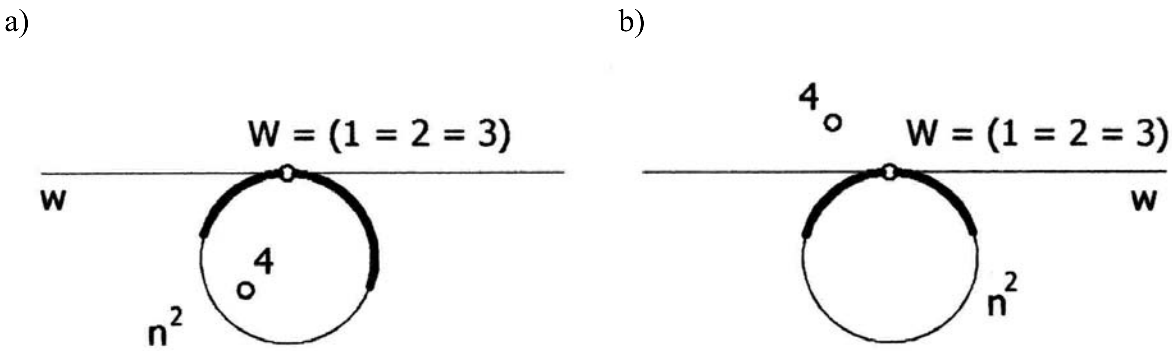

Figure 3.

In case the base quadrangle **1=2=3**,**4** of the pencil of conics P^2 is concave, the elements of this pencil are solely hyperbolas, while the quadrangle is convex, the elements of this pencil are ellipses, hyperbolas and two parabolas.

Let us now check the measure of the angle between the axes of the two parabolas of the pencil P^2 depending on the position of point **4**.

Position of point **4** in relation to the circle $\mathbf{n^2}$ can be as follows:

I. point **4** lies inside the circle $\mathbf{n^2}$,

II. point **4** lies on the circle $\mathbf{n^2}$,

III. point **4** lies outside the circle $\mathbf{n^2}$,

IV. point **4** can be a point at infinity.

Case I. Let point **4** lies inside the circle $\mathbf{n^2}$ (Figure 4).

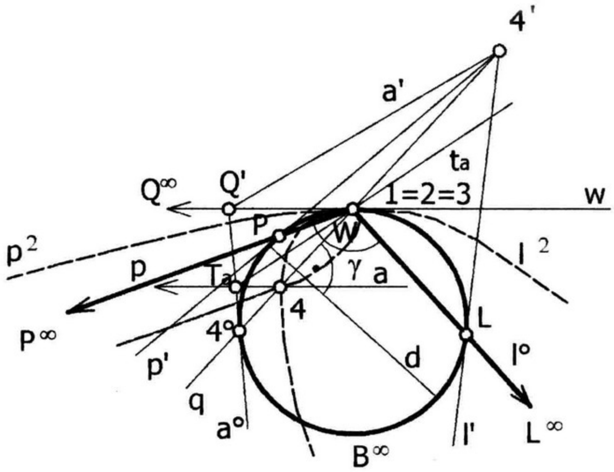

Figure 4.

Let us construct line **a'**, to which corresponds ellipse $\mathbf{a^2}$ with a diameter **AW** in transformation E. Point $\mathbf{4^o}$ on the circle $\mathbf{n^2}$ corresponds to point **4**. Through point **4** passes tangent to the ellipse line **a**, which is parallel to **w**, while through point $\mathbf{4^o}$ passes the tangent $\mathbf{a^o}$ to the circle $\mathbf{n^2}$**.** Lines **a** and $\mathbf{a^o}$ meet in point $\mathbf{T_a}$. Line $\mathbf{t_a}$ joining points $\mathbf{T_a}$ and **W** is an axis of the elation. Lines $\mathbf{a^o}$ and **w** meet in point **Q'** through which parallel to line $\mathbf{t_a}$ passes line **a'**, on which lies point **4'** corresponding to point **4**. Point **4** lies on the opposite side In relation to line **w** and it is a verte of a pencil of lines, which will be transformed into a pencil of conics $P^2_{1=2=3,4}$. Two lines **p'** and **l'** of the pencil of lines are tangent to the circle $\mathbf{n^2}$ in points **P** and **L**. The lines **p**=**WP** and **l**=**WL** are the diameters of two parabolas $\mathbf{p^2}$ and $\mathbf{l^2}$ of the pencil P^2.

Let us draw the diameter *d* of the circle $\mathbf{n^2}$ so that it is perpendicular to the line **q**=$\mathbf{W4^o}$. Points **P** and **L** lie on the same side as point **W** in relation to diameter *d*, and thus $\angle(\mathbf{p},\mathbf{l}) = \gamma > 90^o$ (Figure 4).

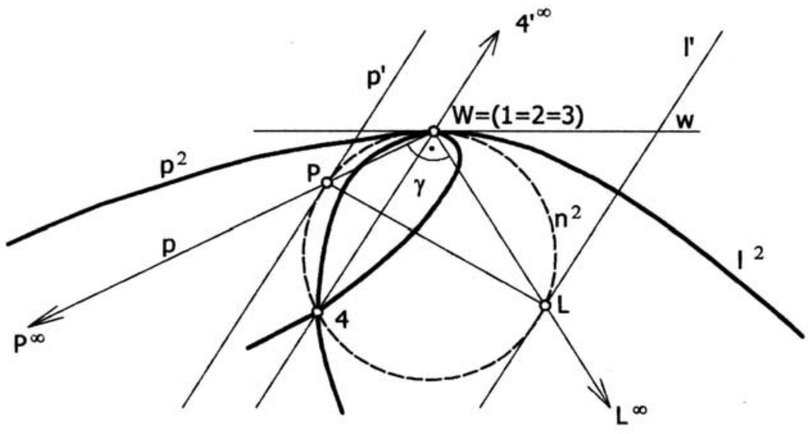

Figure 5.

Case II. If point 4 lies on the circle $\mathbf{n^2}$, then corresponding to it point **4'** lies at infinity. The two tangent lines **p'** and **l'** are parallel and meet the circle at the two endpoints of the diameter *d*. We conclude that $\angle(\mathbf{p},\mathbf{l}) = \gamma = 90^o$ (Figure 5).

Case III. If point **4** lies outside the circle $\mathbf{n^2}$, then corresponding to it point **4'** lies on the same side as **4** in relation to the line **w**. This means that tangent points **P** and **L** of lines **p'** and **l'** with the circle $\mathbf{n^2}$ lie on the opposite side to the point **W** in relation to diameter *d*. We conclude that $\angle(\mathbf{p},\mathbf{l}) = \gamma < 90^{\circ}$ (Figure 6).

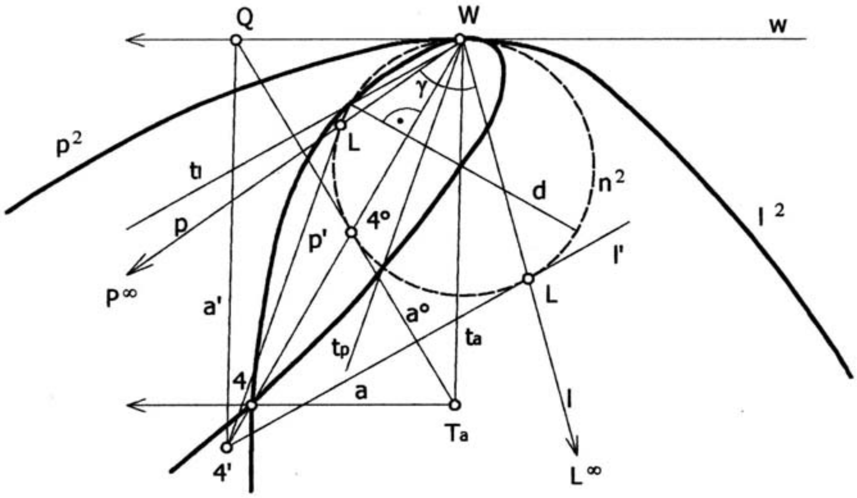

Figure 6

Case IV. If point **4** lies at infinity $\mathbf{4}^{\infty}(P^2_{1=2=3,4\infty})$, then corresponding to it in transformation point **4'** lies on the circle $\mathbf{n^2}$. The tangents **p'** and **l'** drawn from this point to the circle $\mathbf{n^2}$ coincide and thus the points of tangency also coincide (**P**=**L**=**4'**). We conclude that the axes **l** and **p** of the two parabolas coincide, which means that the two parabolas $\mathbf{p^2}$ and $\mathbf{l^2}$ coincide. In this case γ=0 (Figure 7).

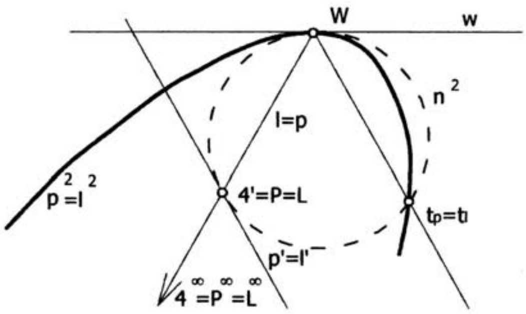

Figure 7

4. Theorem

If we have given a pencil of straight lines **4'**(**a'**, **b'**, **c'**...) and corresponding in elation to it pencil of conics $P^2_{1=2=3,4}(\mathbf{a}^2, \mathbf{b}^2, \mathbf{c}^2 \ldots)$, then these pencils are projective.

Double cross-ratio of four conics $(\mathbf{a}^2\mathbf{b}^2\mathbf{c}^2\mathbf{d}^2) = \mu$ belonging to the pencil $P^2_{1=2=3,4}$ is measured by a double cross-ratio of four straight lines $(\mathbf{abcd})=\mu$, which are the tangent lines to the given conics at point **4**.

In Figure 8 we have drawn a pencil of conics $P^2_{1=2=3,4}$ and we have determined a pencil of perspective straight lines (**4'**).

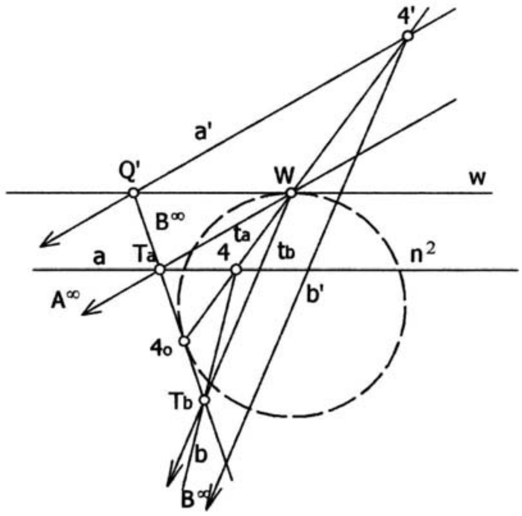

Figure 8

$$4(a,b,c...) \wedge a^{o}(T_a,T_b,T_c...) \wedge W(t_a,t_b,t_c...) \wedge n'^{\infty}(A^\infty,B^\infty,C^\infty...) \wedge 4'(a',b',c'...)$$

The ending elements in this perspective chain are projective as stated. **$4(a,b,c...) \wedge 4'(a',b',c'...)$.**

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PĘKI STOŻKOWE WZAJEMNIE ŚCIŚLE STYCZNYCH $P^2_{1=2=3,4}$

W pracy podano definicję kwadratowego przekształcenia , w którym w dowolnej prostej **a'** przyporządkowujemy taką stożkową a^2, która jest relacyjnym przekształceniem okręgu n^2.

Rozpatrzono trzy przypadki położenia prostej **a'** względem okręgu n^2, które wykazały, iż wszystkie proste nie przechodzące przez środek relacji w przekształcają się w stożkowe wzajemnie ściśle styczne. Sformułowano również twierdzenia dotyczące pęku stożkowych prostych **4'(a',b',c'...)** oraz podporządkowanemu w przekształceniu pękowi stożkowych $P^2_{1=2=3,4}(a^2, b^2, c^2 ...)$, mówiące o rzutowości tych pęków.